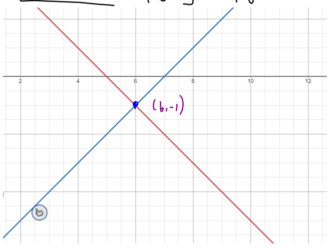
Chapter 1:

$x + y = 5 \rightarrow y = 5 - x$ $\rightarrow 6 + y = 5 \Rightarrow y = -1$

$x - y = 7$ ($y = x - 7$) $\rightarrow 6 - y = 7 \Rightarrow y = -1$



xample 1.

Solve the systems using Gauss-Elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\quad - 3w &= -3 \end{aligned}$$

$x + y = 5$
$2x - y + w = 7$ } Linear

$x^2 - y = 10$ $\quad \sin x + y = e^x$
$e^x + w = 7$ } not linear.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

B is another matrix.

$A + B \quad A - B \quad A \cdot B \quad \frac{A}{B}$

$\frac{2}{3} = 2 \cdot \frac{1}{3} = 2 \cdot 3^{-1}$ (inverse of 3)

$\frac{A}{B} = A \cdot \frac{1}{B} = A \cdot B^{-1}$ → inverse matrix

System

$Ax = B \Rightarrow x = A^{-1} \cdot B$

1.7 Diagonal, Triangular, and Symmetric Matrices 1.8 is not included 1.9 Applications of Linear Systems	Pa Pa Pa

Ch2

Let A be a matrix. Det(A) = number

Cramer's rule

5.1 Eigenvalues and eigenvectors (We cover earlier as some Students taking Mat213 and Mat212 together, just to help them in Mat213)
5.1 Eigenvalues and eigenvectors

3.1 Vectors in 2-Space, 3-Space, and n-Space 3.2 Norm, Dot Product, and Distance in R^n	P: P: P:
3.3 Orthogonality 3.4 is not included 3.5 Cross Product. Linearly independent vectors. Rank and Nullity	P:
3.5 Cross Product. Linearly independent vectors. Rank and Nullity	P:

THEOREM 4.8.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) *A is invertible.*
(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
(c) *The reduced row echelon form of A is I_n.*
(d) *A is expressible as a product of elementary matrices.*
(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.
(g) $\det(A) \neq 0$.
(h) *The column vectors of A are linearly independent.*
(i) *The row vectors of A are linearly independent.*
~~(j) The column vectors of A span R^n.~~
~~(k) The row vectors of A span R^n.~~
~~(l) The column vectors of A form a basis for R^n.~~
~~(m) The row vectors of A form a basis for R^n.~~
(n) *A has rank n.*
(o) *A has nullity 0.*

Chapter 4

The profit is a function of x and y and is given by the *profit function*

$$P = 4x + 6y$$ ✓

Summarizing, we want to maximize the *objective function*

$$P = 4x + 6y \qquad (1)$$ ✓

subject to the condition that x and y must be a solution of the system of constraints:

$$\begin{cases} 2x + y \le 180 & (2) \\ x + 2y \le 160 & (3) \end{cases}$$

Chapter 1

Linear (Algebraic) Equations:

- In two dimensions a **line** in a rectangular *xy*-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a,\ b \text{ not both } 0)$$

This is **a linear equation in the variables *x* and *y***

- In three dimensions a **plan** in a rectangular *xy*-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a,\ b,\ c \text{ not all } 0)$$

This is **a linear equation in the variables *x* , *y an z***

- A **linear equation** in the n variables $x_1, x_2, \dots, x_n$ has the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ and b are real constants. $a_1, a_2, \dots, a_n$ not all zero

- The variables in a linear equation are sometimes called **unknowns**.

- In case $b=0$, the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

is said to be ***homogeneous linear equation in the variables*** $x_1, x_2, \dots, x_n$

$x + 3y = 7,$ ✓ $\quad y = \frac{1}{2}x + 3z + 1,$ → $\frac{1}{2}x - y + 3z + 1 = 1$

$x_1 - 2x_2 - 3x_3 + x_4 = 7$ ✓ $\quad x_1 - 2x_2 = 0$ (homogeneous)

are linear

$x + 3\sqrt{y} = 5,$ ✗ $\quad 3x + 2y - z + xz = 4,$ ✗

and $y = \sin x$

Definition: A **system of linear equation (or a linear system)** is a **finite set of linear equations** in the variables $x_1, x_2, \dots, x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

- **Some definitions:**

- A sequence of numbers $s_1, s_2, \dots, s_n$ is called a **solution** of the system, if when we substitute **x_1 by s_1, x_2 by s_2, …, x_n by s_n the system is satisfied**

- If a system has ***no solution*** is said to be **inconsistent**.

- If there is ***at least one solution*** of the system, it is called **consistent**.

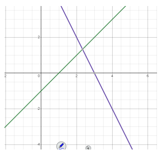
- ***Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions***

one solution — infinitely many

$x - y = 1 \rightarrow y = x - 1$

$2x + y = 6 \rightarrow y = 6 - 2x$

$x - y = 1 \rightarrow y = x - 1$
$+\ 2x + y = 6$
$3x = 7$
$x = 7/3$
$y = \frac{7}{3} - 1$
$y = 4/3$
$(7/3, 4/3)$
unique solution (one solution)



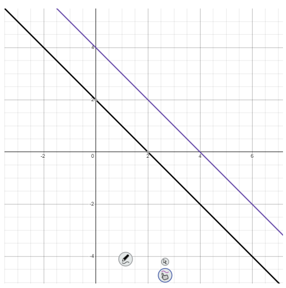
$x + y = 4 \rightarrow 3x + 3y = 12$

$3x + 3y = 6$

disagree - no solution

$y = 4 - 1x$
$3y = 6 - 3x$
$y = 2 - 1x$

$-3x - 3y = -12$
$+\ 3x + 3y = 6$
$0 = -6$!



$4x - 2y = 1 \rightarrow 16x - 8y = 4$

$16x - 8y = 4$

infinitely many

$4x - 1 = 2y$
$2x - \frac{1}{2} = y$

$16x - 4 = 8y$
$2x - \frac{1}{2} = y$

$x = 0$
$y = -1/2$

$4x - 2y = 1$ ✓
$16x - 8y = 4$ ✓
$0 - 8 \cdot -\frac{1}{2} = 4$

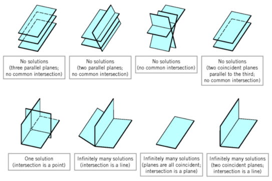
$\Rightarrow y = 2x - \frac{1}{2}$

$x = 0$
$y = -\frac{1}{2}$ ✓

Exercise

$x = 1$, $y = ?$ $\quad x = 1$, $y = ?$ $\quad x = 1907$, $y = ?$

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad \begin{aligned} a_{11}x_1 \qquad\qquad &= b_1 \\ a_{22}x_2 \qquad &= b_2 \\ \vdots \qquad & \vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

Diagonal form
(Same solution eaiser to solve)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ & \vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

Triangular form
(Same solution easier to solve)

System —(elimination algorithm)→ Triangular
→ Diagonal

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

1th column ↓ ; ← 1th row

$1 \cdot x + 1 \cdot y + 0 \cdot z = 9$
$2x + 4y - 3z = 1$
$3x + 6y - 5z = 0$

$$\to \begin{bmatrix} 1 & 1 & 0 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 3 & 3.5 & 1 \\ 6 & 1 & 0 & 7 \\ 2 & 0 & -8 & 0 \end{bmatrix} \qquad \begin{cases} 0.5x + 3y + 3.5z = 1 \\ 6x + y + 0 \cdot z = 7 \\ 2x + 0 \cdot y - 8z = 0 \end{cases}$$

- **Elementary row operations**
  - Multiply an equation (**or a row**) by an nonzero constant
  - Add a multiple (by a constant) of one equation (**or a row**) to another equation (**or a row**)
  - Interchange two equations (**or two rows**)

original

$$\begin{cases} x + y = 0 \\ x - 2y = 6 \end{cases}$$

$2x + 2y = 0$
$+\ x - 2y = 6$
$3x = 6$
$x = 2$

$\to 4 + 2y = 0 \quad y = -2$

add 5 times the first eq. to second one

$x + y = 0$
$6x + 3y = 6$

$-6x - 6y = 0$
$+\ 6x + 3y = 6$
$-3y = 6$
$y = 2$
$x = 2$

Solve

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases} \to \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix}$$

$R2 \to -2R1 + R2$
$R3 \to -3R1 + R3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$R2 \to \frac{1}{2} R2$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$R3 \to -3R2 + R3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

$R3 \to -2R3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

STOP

$x + y + 2z = 9$
$y - \frac{7}{2}z = -\frac{17}{2}$
$z = 3$

$y - \frac{21}{2} = -\frac{17}{2}$
$y = \frac{21}{2} - \frac{17}{2} = 2$
$x + 2 + 6 = 9 \quad x = 1$

$x = 1$
$y = 2$
$z = 3$

DO NOT STOP

$$\begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to$$

$R2 \to \frac{7}{2}R3 + R2$
$R1 \to -2R3 + R1$

$\frac{7}{2} \cdot 3 - \frac{17}{2}$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R1 \to -R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Echelon Forms

Row-echelon form: A matrix is said to be in row-echelon form if it satisfies:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

From 6[th] step of previous examples

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leader 1.
2. In any two successive rows that do not consist entirely of zeros, the leader 1 in the lower row occurs farther to the right than the leader 1 in the higher row.
3. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

Echelon Forms

Reduced row-echelon form: A matrix is said to be in reduced row-echelon form if it is in row-echelon form and satisfies in addition:

4. Each *column* that contains a leader 1 has zeros everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

From last step of previous examples

Note: A matrix in reduced row-echelon form is in row-echelon form. The inverse is not necessary true.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix},$$

1. ✓
2. ✓ REF
3. ✓
4. X not RREF

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

1 ✓
2 ✓
3 ✓ RREF
4 ✓

RREF ⟶ REF ✓

System ⟶ REF / RREF ⇒ Solve

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

*: Arbitrary number

1. ✓
2. ✓ REF
3. ✓
4. X not RREF

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

1. ✓
2. ✓
3. ✓
4. ✓

RREF ✓ REF ✓

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

1. X
2.
3.
4.

Neither REF nor RREF

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

1 ✓
2 X
3
4

neither REF nor RREF

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

0 matrix is in RREF

System ⟶ ↗ R

System --(elimination algorithm)--> A new system

- REF (elimination)
- RREF (Jordan-elimination algorithm)

RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to \begin{cases} x_1 = 3 \\ x_2 = -1 \\ x_3 = 0 \\ x_4 = 5 \end{cases}$$

o unique solution

NO SOLUTION

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$1x_1 + 0x_2 + 0x_3 = 0 \to x_1 = 0$

$x_2 + 2x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 1$

$0 = 1$

0 0 0 . . . . nonzero $\Rightarrow$ no solution.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (ignore)

$1x_1 + 3x_3 = -1$

$1x_2 - 4x_3 = 2$

$0x_1 + 0x_2 + 0x_3 = 0$

$0 = 0$

x_1, x_2 > Basic variables

x_3 Free variable

Solve basic variables in terms of free

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 2 + 4x_3 \\ x_3 \text{ Free} \end{cases}$$

infintely many

$$\begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 0 \end{cases}$$

(c) $\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$1x_1 - 5x_2 + x_3 = 4$

$0x_1 + 0x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

x_1: Basic

x_2, x_3: Free

$$\begin{cases} x_1 = 4 + 5x_2 - x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$$

infinitely many

$$\begin{cases} x_1 = 9 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$$

elimination algorithm

STEP 1. Find the leftmost column which does not consist entirely of zeros.

STEP 2. By interchanging rows if necessary obtain a nonzero entry (*a* say) at the top of the column found in step 1.

STEP 3. Divide the first row by *a* to obtain a leading 1.

STEP 4. Add suitable multiples of the first row to the rows below so that all the entries below the leading 1 become 0.

STEP 5. Ignore the first row of the matrix and repeat the above procedure on the matrix which remains. Continue in this way until the entire matrix is in echelon form.

(ref)

Step6. Starting from the bottom: multiple each row with a suitable constant and add it to the rows above to get **zeros** above the leading 1's.

(rref)

---

which one is TRUE

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

(a) a unique solution ✓

b) no solution

c) infinitely many solution

F d) it is not a linear system False

F e) $x_1 = 0, x_2 = 0, x_3 = 0$ is a solution False

$$\begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \quad R2 \to 1R1 + R2, \quad R3 \to -3R1 + R3$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R2 \to -1 \cdot R2$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R3 \to 10R2 + R3$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad R3 \to -\frac{1}{52} R3$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{REF}$$

$0x_1 + 0x_2 + x_3 = 2$

$x_2 - 5x_3 = -9$, $x_2 - 10 = -9$, $x_2 = 1$

$x_1 + x_2 + 2x_3 = 8$

$x_1 + 5 = 8$, $x_1 = 3$

$$\begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

---

which one is TRUE

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\quad - 3w &= -3 \end{aligned}$$

a) it is not linear False.

b) $x = y = z = w = 1$ is a solution False

c) a unique

(d) infinitely many

e) no solution

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \quad R2: -2R1 + R2, \quad R3: R1 + R3, \quad R4: -3R1 + R4$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \quad R2: \frac{1}{3} R2$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \quad R3 \to -R2 + R3, \quad R4 \to -3R2 + R4$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 + 2x_3 - x_4 = -1$

$x_2 - 2x_3 = 0$

x_1, x_2: Basic

x_3, x_4: Free

Solve the system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 - x_2 - 2x_3 = 1 \\ 3x_1 - x_3 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & -2 & 1 \\ 3 & 0 & -1 & -1 \end{bmatrix} \quad R2: -2R1 + R2, \quad R3: -3R1 + R3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -4 & -1 \\ 0 & -3 & -4 & -4 \end{bmatrix} \quad R2: -\frac{1}{3} R2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & -3 & -4 & -4 \end{bmatrix} \quad R3: 3R2 + R3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to \text{NO SOLUTION}$$

$$\begin{cases} x + 2y = 1 \\ 2x + (a^2-5)y = a-1 \end{cases}$$

For which value of a, does the system have
- a unique
- no solution
- infinitely many

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2-5 & a-1 \end{array}\right]$$

R2 : -2R1+R2

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{array}\right]$$

R2 : $\frac{1}{a^2-9}$ R2

WHAT IF $a^2-9=0$
$a^2=9$
$a=3,-3$

Case 1: $a=3$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow x_1+2x_2=1$
x_1: Basic
x_2: Free

inf. many.

Case 2: $a=-3$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -6 \end{array}\right]$$

NO SOLUTION

Case 3: $a \neq 3, a \neq -3$
For ex: $a=0$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -9 & -3 \end{array}\right]$$

$\rightarrow x_1+2x_2=1$
$\rightarrow -9x_2=-3$

a unique solution

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2-14)z &= a+2 \end{aligned}$$

For what value of a,
a) no solution
b) a unique
c) infinitely many

For what value of a,
a) no solution
b) a unique
c) infinitely many

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

R2 : −3R1+R2
R3 : −4R1+R3

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

R2 : $-\frac{1}{7}$R2

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

R3 : 7R2+R3

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

R3 : $\frac{1}{a^2-16}$R3

$a^2-16=0$, $a^2=16$, $a=4, a=-4$

Case 1: a = 4 : inf. many

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+2y-3z=4$
$y-2z=10/7$
x, y : Basic
z : Free

inf. many

Case 2: a = −4, no solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

$0x+0y+0z=-8$
$0=-8$

Case 3: a ≠ 4, a ≠ −4 : a unique

Let's say a = 5

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 9 & 1 \end{array}\right]$$

$x+2y-3z=4$
$y-2z=\frac{10}{7}$
$9z=1$
$z=\frac{1}{9}$

mcq: For which a, no solution
a) −4 b) −3 c) −2 d) −1 e) 0

Exercise 2: Use Gaussian elimination to find the solution for the given system of equations.

Which one is TRUE
a) no solution
b) a unique
c) inf. many
d) (False) x = y = z = 0 is a solution
e) (False) Exactly 3 solutions

$2x+5y=9$
$x+2y-z=3$
$-3x-4y+7z=1$

$$\left[\begin{array}{ccc|c} 2 & 5 & 0 & 9 \\ 1 & 2 & -1 & 3 \\ -3 & -4 & 7 & 1 \end{array}\right]$$

R1 ⟷ R2

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 9 \\ -3 & -4 & 7 & 1 \end{array}\right]$$

R2 : −2R1+R2
R3 : 3R1+R2

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 10 \end{array}\right]$$

R3 : −2R2+R3

NO SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$0x+0y+0z=4$

What condition, if any, must a, b, and c satisfy for the linear system to be consistent?

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 4 & 0 & c \end{array}\right]$$

R2 : −2R1+R2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 2 & 0 & b-2a \\ 0 & 4 & 0 & c \end{array}\right]$$

R2 : $\frac{1}{2}$R2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 0 & \frac{b-2a}{2} \\ 0 & 4 & 0 & c \end{array}\right]$$

R3 : −4R2+R3

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 0 & \frac{b-2a}{2} \\ 0 & 0 & 0 & -2(b-2a)+c \end{array}\right] = 0$$

$-2(b-2a)+c=0$

$b-2a-\frac{c}{2}=0$

Example 3. Solve the following system for x, y, and z.

$\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1$
$\frac{2}{x}+\frac{3}{y}+\frac{8}{z}=0$
$-\frac{1}{x}+\frac{9}{y}+\frac{10}{z}=5$

I have been killed

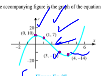
$a=\frac{1}{x}$ $b=\frac{1}{y}$ $c=\frac{1}{z}$

$a+2b-4c=1$
$2a+3b+8c=0$
$-a+9b+10c=5$

elimination alg. → a = ..., b = ..., c = ...

Find x, y, z

Example 4. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y=ax^3+bx^2+cx+d$.



(0,10) : $10=d$

(1,7) : $7=a+b+c+d$

(3,−11) : $-11=27a+9b+3c+d$

(4,−14) : $-14=64a+16b+4c+d$

Linear System
$a+b+c=-3$
$27a+9b+3c=-21$
$64a+16b+4c=-24$

(exercise)

mcq: Find a :
a) b) c) d) e)

- Remarks on homogeneous Linear Systems

A system of linear equations is said to be homogeneous if the constant terms are all zero

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \quad\quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

$x_1 = x_2 = x_3 = 0$ (trivial solution or zero solution)

Homogeneous systems → a unique solution (zero solution); ~~no solution~~; infinitely many solution

Conclusion: Homogeneous systems are always consistent.

Shortcut:

THEOREM 1.2.2 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

Solve

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$$

unknowns: 3
equations: 2 $3 > 2$
infinitely many solutions.

elimination

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \quad R2: -2R1 + R2$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 3 & 0 \end{bmatrix} \quad R2: -\tfrac{1}{5}R2$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -3/5 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 0 \\ x_2 - \tfrac{3}{5}x_3 = 0 \end{cases}$$

x_1, x_2: Basic
x_3: Free
inf. many solutions

---

not homogeneous

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + 4x_2 - 2x_3 = 5 \end{cases}$$

unknowns: 3
equations: 2 $3 > 2$
Mistake: $3 > 2 \Rightarrow$ inf. many.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & -2 & 5 \end{bmatrix} \quad R2: -2R1 + R2$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{no solution}$$

plies only to homogeneous systems—a *nonhomogeneous* system with more unknowns than equations need not be consistent. However, we will prove later that if a nonhomogeneous system with more unknowns then equations is consistent, then it has infinitely many solutions.

not homogeneous + # unknowns > # equations + consistent ⇒ inf. many.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_3 = 8 \end{cases}$$

$3 > 2$.
$x_1 = 2, x_2 = 2, x_3 = -4$
⇒ infinitely many.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -1 & 8 \end{bmatrix} \quad R2: -2R1 + R2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 8 \end{bmatrix} \rightarrow R2: -\tfrac{1}{2}R2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/2 & -4 \end{bmatrix}$$

x_1, x_2: Basic
x_3: Free
inf. many.

Definition and Notation

- A matrix is a rectangular array of numbers. The numbers in the array are called entries
- A general $m{\times}n$ matrix A (it reads m by n matrix A) is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

(Row 1, Row 2, row m; column 1, column 2, column n)

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} or $(A)_{ij}$

Let $A = \begin{bmatrix} 1 & 2 & -5 \\ 0 & 7 & 10 \end{bmatrix}$

Size is 2×3

$a_{11} = 1$ $\quad a_{22} = 7$
$a_{23} = 10$ $\quad a_{21} = 0$

- Two matrices, $m{\times}n$ matrix A and $p{\times}q$ matrix B are equal if they have the same size (m=p and n=q) and their corresponding entries are equal

$a_{ij} = b_{ij}$ for all i and j

ex. $A = \begin{bmatrix} 1 & 5 \\ 0 & 7 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & x+y \\ 0 & x-y \end{bmatrix}$

Let A=B. We are asked $2x + 3y = ?$

a) 9 b) 7 c) 5 d) -1 e) -3

$$\begin{cases} x + y = 5 \\ x - y = 7 \end{cases}$$

$2x = 12$

$2x + 3y$

$12 + -3 = 9$

Operation on matrices

Consider a $m \times n$ matrix A and $p \times q$ matrix B

Sum and difference:

Condition: the two matrices have to have the same size ($m=p$ and $n=q$) then:

The sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.

And the difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Multiply each entry of the matrix A by c

example. $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 2 \end{bmatrix}_{2\times3}$, $B = \begin{bmatrix} -1 & 0 & 5 \\ 10 & 6 & 8 \end{bmatrix}_{2\times3}$

$C = \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}_{2\times2}$

$A+B = \begin{bmatrix} 0 & 2 & 8 \\ 7 & 6 & 10 \end{bmatrix}$

$A + C$ = undefined, we can't.

$2A - 3B = \begin{bmatrix} 2 & 4 & 6 \\ -6 & 0 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 15 \\ 30 & 18 & 24 \end{bmatrix} =$

$= \begin{bmatrix} 5 & 4 & -9 \\ -36 & -18 & -20 \end{bmatrix}$

We deduce the following condition for the multiplication:

- Condition: Number of columns of A has to be equal to the number of rows of B ($p=q=r$). The obtained matrix is a $m \times n$ matrix

$$A_{m\times r} B_{r\times n} = (AB)_{m\times n} = C_{m\times n}$$

$$AB = D = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & a_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & d_{ij} & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$d_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$$

ex. Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & -2 \end{bmatrix}_{2\times3}$

$1 \cdot 1 + 2 \cdot 0 = 1$
$1 \cdot 0 + 2 \cdot -1 = -2$
$1 \cdot 5 + 2 \cdot -2 = 1$

$-3 \cdot 1 + 0 \cdot 0 = -3$
$-3 \cdot 0 + 0 \cdot -1 = 0$
$-3 \cdot 5 + 0 \cdot -2 = -15$

$\underset{2\times2}{A}\ \underset{2\times3}{B} = \begin{bmatrix} 1_{(1,1)} & -2_{(1,2)} & 1_{(1,3)} \\ -3_{(2,1)} & 0_{(2,2)} & -15_{(2,3)} \end{bmatrix}_{2\times3}$

$\underset{2\times3}{B}\ \underset{2\times2}{A}$ does not exist $\qquad AB \neq BA$

---

mcq: Let $A = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 5 & 10 & 1907 \end{bmatrix}_{2\times4}$

(2,3) entry:
$-14 + 5 + 0 + 0 = -9$

$B = \begin{bmatrix} 1 & 0 & -7 & 9 \\ 2 & 3 & 1 & 0 \\ -2 & -3 & 0 & 5 \\ 7 & 7 & 0 & 7 \end{bmatrix}_{4\times4}$

$2 \cdot -7 + 5 \cdot 1 + 10 \cdot 0 + 1907 \cdot 0 = -14 + 5 = -9$

What is (2,3) entry in AB = ?
a) -9 b) -7 c) 10 d) 1 e) 12

$A_{2\times4} B_{4\times4} = \begin{bmatrix} - & - & - & - \\ - & - & -9 & - \end{bmatrix}_{2\times4}$

---

numbers

$\frac{2}{7} = 2 \cdot \frac{1}{7} = 2 \cdot 7^{-1}$ → inverse of 7

7 and $\frac{1}{7}$ are inverses of each other.
Because $7 \cdot \frac{1}{7} = 1$

Let A and B be two square matrices (#rows = #colms)

$\frac{A}{B} = A \cdot \frac{1}{B} = A B^{-1}$ Inverse of B.

What is B^{-1} = ? $\qquad 3 \cdot \frac{1}{3} = \frac{1}{3} 3 = 1$

$A \cdot B = B \cdot A = I$

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$ → diagonal

- **Invertible Matrix**
- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* and B is called an *inverse* of A.
- If no such matrix B can be found, then A is said to be *singular*.

Remark:
invertible: A^{-1} exists.
Singular: A^{-1} does not exist.

...mple 1: Verify if B is the inverse of A

$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ $\qquad$ Verify $A^{-1} = B$ or $B^{-1} = A$

...tion:

$\underset{2\times2}{A}\ \underset{2\times2}{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\Rightarrow A^{-1} = B.$

$\underset{2\times2}{B}\ \underset{2\times2}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\quad B^{-1} = A.$

Next time: How do we calculate A^{-1}.

1. Adjoin the identity matrix to the right side of A, thereby producing a matrix of the form $[A \mid I]$

2. Apply the reduced-row echelon process to A until the left side is reduced to I; this process will convert the right side to A^{-1}, so that the final matrix will have the form $[I \mid A^{-1}]$

$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. Calculate A^{-1}!

$\left[\begin{array}{cc|cc} 2 & -5 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} -1 & 3 & 0 & 1 \\ 2 & -5 & 1 & 0 \end{array}\right]$

$R1: -1R1$ $\left[\begin{array}{cc|cc} 1 & -3 & 0 & -1 \\ 2 & -5 & 1 & 0 \end{array}\right]$ $R2: -2R1+R2$

$\left[\begin{array}{cc|cc} 1 & -3 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right]$ $R1: 3R2+R1$

$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right]$ → $A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Calculate A^{-1}.

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R2: -2R1+R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$

R2: 2·R1

STOP SAY
A is singular
(no inverse)
A^{-1} does not exist.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Calculate A^{-1}.

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$ $R2: -2R1+R2$, $R3: -R1+R3$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$ $R3: 2R2+R3$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$ $R3: -R3$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ $R2: 3R3+R2$, $R1: -3R3+R1$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ $R1: -2R2+R1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$. which one is TRUE.

3x3

A) it is not square False
B) it is 4x3 matrix False
C) (2,2) entry in A is -1 False
d) A is singular (circled)
e) (1,1) entry in A^{-1} is 2.

$A = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$ $R2: -2R1+R2$, $R3: 1R1+R3$

$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$

$-8R_2+R_3$: 0 0 0

no inverse.

R3 = -1·R2

conclusion: if you notice two rows multiple of each other, say no inverse!

---

why do we care A^{-1}!

We can solve system by using inverse matrices.

Writing a system in matrix form

We can rewrite every system as $Ax=b$ (matrix form)

$A \, x = b$

Coefficient matrix, unknown vector, Right-hand side

ex Rewrite $\begin{cases} x_1 - x_2 = 5 \\ 2x_1 + 5x_2 = 10 \end{cases}$ in matrix form.

$A \, x = b$

$\begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

2x2, 2x1

$\begin{bmatrix} x_1 - x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$x_1 - x_2 = 5$
$2x_1 + 5x_2 = 10$

$\begin{cases} x_1 - 2x_2 + 3x_3 = 10 \\ 2x_1 + 5x_2 - x_3 = 7 \\ 3x_1 + 7x_2 - x_3 = 11 \end{cases}$

$A \cdot x = b$

$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & -1 \\ 3 & 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 11 \end{bmatrix}$

3x3, 3x1

$A^{-1} \cdot A x = A^{-1} b$

$$x = A^{-1} \cdot b$$

examples: (Next time)

$A^{-1} A x = A^{-1} b$

$I \cdot x = A^{-1} b$

$\boxed{x = A^{-1} \cdot b}$

We are given $\begin{cases} 2x_1 - 5x_2 = 2 \\ -x_1 + 3x_2 = -1 \end{cases}$

a) Solve by elimination (exercise) ✓
b) Rewrite the system in matrix form $Ax = b$.
c) Calculate A^{-1}
d) Solve the system by $x = A^{-1} \cdot b$

---

b) $A \cdot x = b$

$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

c) $A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ Please check lecture notes.

d) $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \cdot b = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (2x2, 2x1)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{x_1 = 1, \; x_2 = 0}$ o unique solution.

---

example. $\begin{cases} x_1 + 2x_2 = -2 \\ 2x_1 + 4x_2 = -4 \end{cases}$

a) Rewrite in the form $Ax = b$
b) Calculate A^{-1}.
c) Solve the system by $x = A^{-1} \cdot b$
d) Solve by elimination

a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$A \cdot x = b$

b) A^{-1} does not exist. See Lecture notes.

c) $x = A^{-1} \cdot b$

Do not say: NO SOLUTION.
Say: This method does not work.

d) $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \end{bmatrix} \to R2: -2R1 + R2$

$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to x_1 + 2x_2 = -2$

x_1: Basic x_2: Free

$\begin{cases} x_1 = -2 - 2x_2 \\ x_2 \text{ free} \end{cases}$ inf. many solutions.

---

mcq: We are given $\begin{cases} x_1 + 2x_2 = -2 \\ x_2 + 4x_2 = -4 \end{cases}$.
which one is False.

True a) We can put it into matrix form $Ax = b$
True b) We can solve it by elimination
False c) If we write in $Ax = b$, A is invertible $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
True d) We can't solve it by $x = A^{-1} \cdot b$
True e) We have infinitely many solutions.

We are given: $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = -1 \\ x_1 + 8x_3 = 1 \end{cases}$.

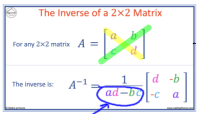
a) Rewrite in $Ax = b$.
b) Calculate A^{-1} (if possible)
c) Solve by $x = A^{-1} b$
d) Solve by elimination (exercise)

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

b) $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$ (Lecture notes)

c) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$ (3x3, 3x1)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$



Ch2. Determinant.

If A is an $n \times n$ matrix, then the following statements are equivalent, that is all true or all false. (zero solution = $x_1 = x_2 = \dots x_n = 0$)

1. A is invertible
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
3. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
4. The reduced row-echelon form of A is I_n

if 1 is true: $x = A^{-1} \cdot 0 = 0$. $x = A^{-1} b$ =

$[A \mid I] \to [I \mid A^{-1}]$

Let A be 3x3 matrix. We are given Ax = 0 has only zero solution. Then which one is false

example. Let A be a 3x3

a) A^{-1} exists
b) A is a square matrix
c) RREF of A is I

False e) $Ax = b$ has infinitely many solutions.

- A is invertible — must be True
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution — True
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$ — must be true
- The reduced row-echelon form of A is I_n — must be true

- **Transpose of a matrix:**

If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

That is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth

$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 5 \end{bmatrix}_{2\times 3}$, $B^T = \begin{bmatrix} 1 & -3 \\ 2 & 0 \\ 3 & 5 \end{bmatrix}_{3\times 2}$

If A is $m\times n$, A^T is $n \times m$.

- **Proprieties of transpose and inverse**
- $I^T = I$ ✓
- $((A^T)^T = A$
- $(A + B)^T = A^T + B^T$ and $(A - B)^T = A^T - B^T$ ✓
- $(kA)^T = kA^T$, where k is any scalar
- $(AB)^T = B^TA^T \neq A^T B^T$ (Give an example: exercise)
- If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

$A = \begin{bmatrix} 1 & 2 \\ 6 & 9 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{-3}\begin{bmatrix} 9 & -2 \\ -6 & 1 \end{bmatrix} \Rightarrow (A^{-1})^T = \frac{1}{-3}\begin{bmatrix} 9 & -6 \\ -2 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 6 \\ 2 & 9 \end{bmatrix} \Rightarrow (A^T)^{-1} = \frac{1}{-3}\begin{bmatrix} 9 & -6 \\ -2 & 1 \end{bmatrix}$

$(A^T)^{-1} = (A^{-1})^T$

- **Trace of a square matrix A:**

-If A is a square matrix, then the trace of A, denoted by $tr(A)$, is defined to be the sum of the entries on the main diagonal of A.

-The trace of A is not defined if A is not a square matrix.

For an $n \times n$ matrix $A = [a_{ij}]$,

Sum of diagonal entries: $= tr(A) = \sum_{i=1}^{n} a_{ii}$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & 7 \\ 10 & 15 & -9 \end{bmatrix}_{3\times 3}$ $tr(A) = 1 + 5 + -9 = -3$.

$B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & 10 \end{bmatrix}_{2\times 3}$ $tr(B)$ does not exist

Proprieties of transpose and inverse

$I^T = I$ $I = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \Rightarrow I^T = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$ ✓

$((A^T)^T = A$ ✓

$(A + B)^T = A^T + B^T$ and $(A - B)^T = A^T - B^T$

$(kA)^T = kA^T$, where k is any scalar

$(AB)^T = B^T A^T$ $(\neq A^T B^T)$

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

$(A^T)^{-1} = \frac{1}{1}\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

$(A^{-1})^T = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$

- **Commutativity:**

  The multiplication of matrices is not commutative. That is

  $AB \neq BA$ in general

- **Cancelation:**

$A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 12 \\ 14 & 25 \end{bmatrix}$ $BA = \begin{bmatrix} 11 & . \\ . & . \end{bmatrix}$ $AB \neq BA$

- **Cancelation:**

  The cancellation law is not valid for matrix multiplication.

That is

If $AB=AC$ (with $A \neq 0$) this doesn't necessary implies that $B=C$

For numbers: $3x = 3y \Rightarrow x = y$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

? $AB = AC$

$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \Rightarrow B \neq C$

Zero divisors: $AB = 0$ doesn't necessary mean $A = 0$ or $B = 0$

numbers: if $xy = 0 \Rightarrow x = 0$ or $y = 0$.

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$ ⟶ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ neither is 0.

$(AB)^{-1} = B^{-1}A^{-1}$ $(\neq A^{-1}B^{-1})$

$A^0 = I$. $A^n = \underbrace{A.A.\dots.A}_{n \text{ times}}$

$3^{-2} = (3^{-1})^2 = 3^{-1} \cdot 3^{-1}$ (3^{-1}: inverse of 3) $A^{-n} = (A^{-1})^n = \underbrace{A^{-1}.A^{-1}\dots A^{-1}}_{n \text{ times}}$

Theorem 1

If A is a square matrix and r and s are integers, then

$$A^r A^s = A^{r+s}, \qquad (A^r)^s = A^{rs}$$ ✓ ✓

Theorem 2

If A is an invertible matrix, then: $(2^{-1})^3 = (2^3)^{-1} = 2^{-3}$

- A^{-1} is invertible and $(A^{-1})^{-1} = A$ ✓
- A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \dots$ ✓
- For any nonzero scalar k, the matrix kA is invertible and

  $(kA)^{-1} = (1/k)A^{-1}$ ($1/k$: inverse of k) ✓

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

Calculate:
a) A^{-1}
b) A^2
c) A^3
d) A^{-2}
e) A^{-3}
f) $A^0 = ?$

a) $A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

b) $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$

c) $A^3 = A^2 \cdot A^1 = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$

d) A^{-2}

$(A^{-1})^2$
$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$

$(A^2)^{-1}$
$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}^{-1}$
$\frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$

e) (exercise) A^{-3}

$(A^3)^{-1}$ $(A^{-1})^3$

f) $A^0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

---

mcq: Let A and B be 5x5 invertible matrices. which one is False

a) $(AB)^{-1} = B^{-1}A^{-1}$ ✓
b) $(A^T)^{-1} = (A^{-1})^T$ ✓
c) $(5A)^{-1} = 5A^{-1}$ False $(5A)^{-1} = \frac{1}{5}A^{-1}$
✓ d) $(A+B)^T = A^T + B^T$
✓ e) $B^0 = I$

which one is True

a) if AB is $\underline{0} \Rightarrow$ either A or B must be zero
b) if $AB = AC \Rightarrow B = C$ ✗
c) $A^2 \cdot A^3 = A^{3/2}$ ✗ $A^3 = A.A.A$
d) $A^{-3} = (A^3)^{-1}$ True
e) A^0 can't be I. ✗

1.1 – 1.6 completed ✓

Quizzes: 6pm – 7pm

midterms: 6pm – 7:30pm

Definition `1 A square matrix in which all the entries off of the main diagonal are zero is said to be a ***diagonal matrix***.

Examples of diagonal matrices: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}2&0&0&0\\0&4&0&0\\0&0&-3&0\\0&0&0&8\end{bmatrix}$, $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Diagonal matrices make matrix arithmetic easier.

Diagonal Matrix Arithmetic

Powers

For example, if $D = \begin{bmatrix}d_1&0&\cdots&0\\0&d_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&0&d_n\end{bmatrix}$ is an $n \times n$ diagonal matrix and k is an integer,

then

$$D^k = \begin{bmatrix}{d_1}^k&0&\cdots&0\\0&{d_2}^k&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&0&{d_n}^k\end{bmatrix} \text{ and } D^{-1} = \begin{bmatrix}{d_1}^{-1}&0&\cdots&0\\0&{d_2}^{-1}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&0&{d_n}^{-1}\end{bmatrix}$$

s, if $A = \begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}$, t $\Rightarrow A^2 = \begin{bmatrix}4&0&0\\0&9&0\\0&0&16\end{bmatrix}$

$A^3 = \begin{bmatrix}8&0&0\\0&27&0\\0&0&64\end{bmatrix}$ $A^{-1} = \begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{3}&0\\0&0&\frac{1}{4}\end{bmatrix}$

$B = \begin{bmatrix}3&0&0\\0&5&0\\0&0&0\end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{5}&0\\0&0&\frac{1}{0}\end{bmatrix}$?

B^{-1} does not exist.

---

mcq: Let $B = \begin{bmatrix}-\frac{1}{2}&0&0\\0&5&0\\0&0&-7\end{bmatrix}_{3\times 3}$.

Which one is False.

a) B is diagonal ✓

b) B is invertible ✓

c) (2,2) entry in B^{-1} is $\frac{1}{5}$ ✓ $B^{-1} = \begin{bmatrix}-2&0&0\\0&\frac{1}{5}&0\\0&0&-\frac{1}{7}\end{bmatrix}$

d) B is a 3x3 matrix ✓

e) $Bx = \begin{bmatrix}-2\\7\\10\end{bmatrix}$ has infinitely many solutions. False.

$B^{-1}Bx = B^{-1}\cdot\begin{bmatrix}-2\\7\\10\end{bmatrix}$

$x = B^{-1}\cdot\begin{bmatrix}-2\\7\\10\end{bmatrix}$

$\begin{bmatrix}-\frac{1}{2}&0&0\\0&5&0\\0&0&-7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\7\\10\end{bmatrix}$

$\begin{bmatrix}-\frac{1}{2}x_1\\5x_2\\-7x_3\end{bmatrix} = \begin{bmatrix}-2\\7\\10\end{bmatrix}$

$-\frac{1}{2}x_1 = -2$

$5x_2 = 7$

$-7x_3 = 10$

Definition 2 A square matrix A such that $A^T = A$ is said to be a symmetric matrix.

$A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

$A^T = \begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$

$a_{12} = a_{21}$

$a_{13} = a_{31}$ $\Rightarrow a_{ij} = a_{ji}$

$a_{32} = a_{23}$

$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

---

example. Let $A = \begin{bmatrix}1&x-2&y^3\\5&2&z+5\\1&0&3\end{bmatrix}$

Let A be a symmetric matrix. $x+y+z$ is

$x-2=5 \Rightarrow x=7$

$y^3=1 \Rightarrow y=1$

$z+5=0 \Rightarrow z=-5$.

$x+y+z=3$

a) 1 b) 3 c) 5 d) −4 e) −10

Theorem 1 Let A and B be symmetric matrices with the same size and k any scalar.

Then

1. A^T is symmetric ✓
2. $A+B,\ A-B$ are symmetric ✓
3. kA is symmetric ✓

$A = \begin{bmatrix}3&8&-2\\8&5&0\\-2&0&1\end{bmatrix}$

$2A = \begin{bmatrix}6&16&-4\\16&10&0\\-4&0&2\end{bmatrix}$ $A^T = \begin{bmatrix}3&8&-2\\8&5&0\\-2&0&1\end{bmatrix}$

$0\cdot A = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

Theorem 2 The product of two symmetric matrices A and B is symmetric if and only if $AB = BA$.

$A = \begin{bmatrix}5&1\\1&-2\end{bmatrix}$ $B = \begin{bmatrix}0&4\\4&-2\end{bmatrix}$

A, B are symmetric. $AB = \begin{bmatrix}4&18\\-8&8\end{bmatrix}$ is n[ot] symmet[ric]

Theorem 3 If A is an invertible symmetric matrix then A^{-1} is symmetric.

$A = \begin{bmatrix}2&6\\6&1\end{bmatrix}$ ✓ $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&\\-c&\end{bmatrix}$

$A^{-1} = \frac{1}{-34}\begin{bmatrix}1&-6\\-6&2\end{bmatrix} \Rightarrow A^{-1}$ is symmetric.

Theorem 4 If A is an matrix, then AA^T and A^TA are both symmetric matrices.

$A = \begin{bmatrix}1&2\\3&4\end{bmatrix}$ $AA^T = \begin{bmatrix}5&11\\11&25\end{bmatrix}$

$A^T = \begin{bmatrix}1&3\\2&4\end{bmatrix}$

Theorem 6 If A is invertible, then AA^T and A^TA are both invertible matrices.

exercise: set-up an example.

Triangular Matrices

Definition 4 A square matrix where all the entries below the main diagonal are zero is said to be upper triangular. A square matrix where all of the entries above the main diagonal are zero is said to be a lower triangular matrix.

***Upper triangular* 3 x 3**

$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&0&a_{33}\end{bmatrix}$

***Lower triangular* 3 x 3**

$\begin{bmatrix}a_{11}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

More generally an $n \times n$ matrix $A = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ for all $i > j$ and lower triangular if $a_{ij} = 0$ for all $j > i$.

Theorem 7 Let A be an $n \times n$ matrix.

1. If A is upper triangular, then A^T is lower triangular. ✓
2. If A is lower triangular, then A^T is upper triangular. ✓
3. The product of upper triangular matrices is upper triangular and the product of lower triangular matrices is lower triangular. ✓
4. A triangular matrix is invertible if and only if all of its diagonal entries are nonzero.
5. The inverse of a lower triangular matrix is a lower triangular matrix and the inverse of an upper triangular matrix is an upper triangular matrix.

$A = \begin{bmatrix}12&5&6\\0&-6&7\\0&0&2\end{bmatrix}$ $A^T = \begin{bmatrix}12&0&0\\5&-6&0\\6&-7&2\end{bmatrix}$

mcq: Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 3 & 2 & -1 & 0 \\ 5 & 4 & -7 & 10 \end{bmatrix}$

which one is False

$A^T = \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 5 & 2 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

✓ a) A is lower triangular

✓ b) A^T is upper triangular

✓ c) A is invertible

✓ d) A^{-1} is lower triangular

(e) A is singular

1.8 is not included.

Next time: 1.9

1.8 is not included.

1.9

▶ EXAMPLE 1 **Network Analysis Using Linear Systems**

Figure 1.9.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

Solution As illustrated in Figure 1.9.2, we have assigned arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signaled by a negative value for the flow rate when we solve for the unknowns.

It follows from the conservation of flow at node A that

$$x_1 + x_2 = 30$$

Similarly, at the other nodes we have

$x_2 + x_3 = 35$ (node B)
$x_3 + 15 = 60$ (node C)
$x_1 + 15 = 55$ (node D)

$x_2 = -10$. $x_3 = 45$ $x_1 = 40$

These four conditions produce the linear system

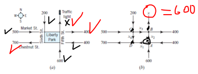
$$\begin{aligned} x_1 + x_2 \quad &= 30 \\ x_2 + x_3 &= 35 \\ x_3 &= 45 \\ x_1 \quad &= 40 \end{aligned}$$

▶ EXAMPLE 2 **Design of Traffic Patterns**

The network in Figure 1.9.3 shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?



Solution (a) If, as indicated in Figure 1.9.3b, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$
Flowing out: $x + 700 + 400$

$x + 1100 = 1700$
$x = 600$.

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.

b)

Solution (b) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	=	$x_1 + x_2$
B	$x_2 + x_3$	=	$400 + x$ = 600
C	$500 + 200$	=	$x_3 + x_4$
D	$x_1 + x_4$	=	700

$x_1 + x_2 = 1000$
$x_2 + x_3 = 1000$
$x_3 + x_4 = 700$
$x_1 + x_4 = 700$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \end{bmatrix} \quad R4 = -R1 + R4$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \end{bmatrix} \quad R4 = R2 + R4$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 1 & 1 & 700 \end{bmatrix} \quad R4 = -R3 + R4$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 1000$
$x_2 + x_3 = 1000$
$x_3 + x_4 = 700$
x_4 is free

$x_3 = 700 - x_4$
$x_2 = 300 + x_4$
$x_1 = 700 - x_4$
x_4 free

$x_2 + 700 - x_4 = 1000$ (300)
$x_1 + 300 + x_4 = 1000$ (700)

$0 \le x_4 \le 700$

the ranges

$0 \le x_1 \le 700$, $300 \le x_2 \le 1000$, $0 \le x_3 \le 700$, $0 \le x_4 \le 700$

Balance the chemical equation

x_1 HCl + x_2 Na_3PO_4 ⟶ x_3 H_3PO_4 + x_4 NaCl
[hydrochloric acid] + [sodium phosphate] ⟶ [phosphoric acid] + [sodium chloride]

3 1 1 3

$1x_1 = 3x_3$	Hydrogen (H)	$x_1 - 3x_3 = 0$
$1x_1 = 1x_4$	Chlorine (Cl)	$x_1 - x_4 = 0$
$3x_2 = 1x_4$	Sodium (Na)	$3x_2 - x_4 = 0$
$1x_2 = 1x_3$	Phosphorus (P)	$x_2 - x_3 = 0$
$4x_2 = 4x_3$	Oxygen (O)	$4x_2 - 4x_3 = 0$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{bmatrix} \quad R2 = -R1 + R2$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{bmatrix} \quad R2 \leftrightarrow R3$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{bmatrix} \quad R2 \leftrightarrow R4$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{bmatrix} \quad \begin{aligned} R4 &= -3R2 + R4 \\ R5 &= -4R2 + R5 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R3 = \frac{1}{3}R3$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R4 = -3R3 + R4$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3x_3 = 0$
$x_2 - x_3 = 0$
$x_3 - \frac{1}{3}x_4 = 0$
x_4 free

$x_3 = \frac{1}{3}x_4$
$x_2 = x_3 = \frac{1}{3}x_4$
$x_1 = 3x_3 = 3 \cdot \frac{1}{3}x_4$
x_4 free

$x_1 = \frac{3}{4}x_4$
$x_2 = \frac{1}{4}x_4$
$x_3 = \frac{1}{4}x_4$
x_4 free

$x_4 = 3$
$x_3 = 1$
$x_2 = 1$
$x_1 = 3$

Sunday: will finish 1-9
Monday: will finish 1-9
Tuesday: EXAM
Wednesday: I will solve exam questions.
Thursday: I will solve exam questions

Polynomial Interpolation

An important problem in various applications is to find a polynomial whose graph passes through a specified set of points in the plane; this is called an ***interpolating polynomial*** for the points. The simplest example of such a problem is to find a linear polynomial

$$p(x) = ax + b \quad (8)$$

whose graph passes through two known distinct points, (x_1, y_1) and (x_2, y_2), in the xy-plane (Figure 1.9.10). You have probably encountered various methods in analytic geometry for finding the equation of a line through two points, but here we will give a method based on linear systems that can be adapted to general polynomial interpolation.

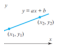


Figure 1.9.10

The graph of (8) is the line $y = ax + b$, and for this line to pass through the points (x_1, y_1) and (x_2, y_2), we must have

$$y_1 = ax_1 + b \quad \text{and} \quad y_2 = ax_2 + b$$

Therefore, the unknown coefficients a and b can be obtained by solving the linear system

$$ax_1 + b = y_1$$
$$ax_2 + b = y_2$$

THEOREM 1.9.1 Polynomial Interpolation

Given any n points in the xy-plane that have distinct x-coordinates, there is a unique polynomial of degree $n-1$ or less whose graph passes through those points.

Find a cubic polynomial whose graph passes through the points

$$(1, 3), \quad (2, -2), \quad (3, -5), \quad (4, 0)$$

x y, x y, x y, x y

Solution Since there are four points, we will use an interpolating polynomial of degree $n = 3$. Denote this polynomial by

$$y = p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$\begin{cases} 3 = a_0 + a_1 + a_2 + a_3 \\ -2 = a_0 + 2a_1 + 4a_2 + 8a_3 \\ -5 = a_0 + 3a_1 + 9a_2 + 27a_3 \\ 0 = a_0 + 4a_1 + 16a_2 + 64a_3 \end{cases}$$

(exercise)

$a_0 = 4, a_1 = 3, a_2 = -5, a_3 = 1.$

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$y = 4 + 3x - 5x^2 + x^3$$

EXAMPLE 7 Approximate Integration

There is no way to evaluate the integral

$$y = \sin\left(\frac{\pi}{2}x^2\right) = f(x) \qquad \int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$$

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

e interval [0, 1] into four equally spaced subintervals (Figure 1.9.

$$f(x) = \sin\left(\frac{\pi x^2}{2}\right) = y$$

s are approximately

$f(0) = 0, \quad f(0.25) = 0.098017, \quad f(0.5) = 0.382683,$
$f(0.75) = 0.77301, \quad f(1) = 1$

(0,0) (0.25, 0.09) (0.5, 0.38) (0.75, 0.77) (1,1)

(exercise)

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

$$p(x) = 0.098796x + 0.762356x^2 + 2.14429x^3 - 2.00544x^4$$

$$\int_0^1 p(x)\,dx \cong 0.43$$

APPLY IT

10. A zoo veterinarian can purchase animal food of four different types: A, B, C, and D. Each food comes in the same size bag, and the number of grams of each of three nutrients in each bag is summarized in the following table:

① Set up the system
② Solve it

		Food			
		A	B	C	D
	N_1	5	5	10	5
Nutrient	N_2	10	5	30	10
	N_3	5	15	10	25

$x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$

For one animal, the veterinarian determines that she needs to combine the bags to get 10,000 g of N_1, 20,000 g of N_2, and 20,000 g of N_3. How many bags of each type of food should she order?

a) $$\begin{cases} 5a + 5b + 10c + 5d = 10 \\ 10a + 5b + 30c + 10d = 20 \\ 5a + 15b + 10c + 25d = 20 \end{cases}$$

b) $$\begin{bmatrix} 5 & 5 & 10 & 5 & 10 \\ 10 & 5 & 30 & 10 & 20 \\ 5 & 15 & 10 & 25 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 10 & 5 & 30 & 10 & 20 \\ 5 & 15 & 10 & 25 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & -5 & 10 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 20 & 20 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 \end{bmatrix} \to \begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ x_2 - 2x_3 = 0 \\ x_3 + x_4 = \frac{1}{2} \end{cases}$$

$x_3 = \frac{1}{2} - x_4$

$x_2 = 2x_3 = 2\left(\frac{1}{2} - x_4\right) = 1 - 2x_4$

$x_1 = 2 - x_2 - 2x_3 - x_4$

$x_1 = 2 - (1 - 2x_4) - 2\left(\frac{1}{2} - x_4\right) - x_4$

$x_1 = 2 - 1 + 2x_4 - 1 + 2x_4 - x_4$

$x_1 = 3x_4$

$x_1 = 3x_4$
$x_2 = 1 - 2x_4$
$x_3 = \frac{1}{2} - x_4$

$$\begin{cases} x_1 = 3x_4 \\ x_2 = 1000 - 2x_4 \\ x_3 = 500 - x_4 \\ x_4 \text{ free} \end{cases}$$

$$\begin{cases} x_1 = 300 \\ x_2 = 800 \\ x_3 = 400 \\ x_4 = 100 \end{cases}$$

---

12 — 3 Tuesday
office hours.

1) Let $K = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 6 \\ 3 & -2 & 4 \end{bmatrix}$. Then calculate trace(2K) + trace(- K) ?

A) 2
B) 5 ✓
C) 7
D) Can not be calculated
E) 3

$10 + -5 = 5$

$2K = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 0 & 12 \\ 6 & -4 & 8 \end{bmatrix}$

$-K = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & -6 \\ -3 & 2 & -4 \end{bmatrix}$

8) Let $M = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 6 \\ 3 & -2 & 4 \end{bmatrix}$. Then calculate trace(3M) + trace(- M) ?

A) Can not be calculated
B) 10 ✓
C) 12
D) 7
E) 8

$15 + -5 = 10$

$3M = \begin{bmatrix} 3 & 6 & 0 \\ 6 & 0 & 18 \\ 9 & -6 & 12 \end{bmatrix}$

$-M = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & -6 \\ -3 & 2 & -4 \end{bmatrix}$

---

2) Which one is **false** ?

A) Multiplyig a row by -5 is a row operation
B) A linear system may have only trivial (zero) solution
C) Not every square matrix has an inverse
D) Dividing a row by 4 is a row operation
E) If a matrix is in row echelon form, it must be in reduced row echelon form

10) Which one is **false** ?

A) If a matrix is in row echelon form, it must be in reduced row echelon form
B) Dividing a row by 2 is a row operation
C) Not every square matrix has an inverse
D) A linear system may have no solution
E) Multiplyig a row by -3 is a row operation

---

3) What can we say about the solution to the following system ?

$$\begin{cases} x + y + z = 0 \\ x - y - z = 2 \\ x + y + 2z = 0 \end{cases}$$

A) One zero solution and one nonzero solution
B) Only zero solution
C) Infinitely many solutions
D) A unique nonzero solution
E) No solution

6) What can we say about the solution to the following system ?

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 2 \\ x_1 + x_2 + 2x_3 = 0 \end{cases}$$

A) Infinitely many solutions
B) A unique nonzero solution
C) No solution
D) Only zero solution
E) One zero solution and one nonzero solution

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$x + y + z = 0$ $x = 1$
$-2y - 2z = 2$ $y = -1$
$z = 0$

Unique solution $\begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases}$

---

4) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 7 & 3 \\ 2 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -5 \\ 6 \end{bmatrix}$. Then which one is **true** ?

A) (2, 1) entry in AB is - 24 ✗
B) B is a 1 x 3 matrix ✗
C) AB is a 1 x 3 matrix ✗
D) BA is a 3 x 1 matrix ✗
E) (1, 2) entry in B^TA is - 11

5) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 7 & 3 \\ 2 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -5 \\ 6 \end{bmatrix}$. Then which one is **true** ?

A) AB is a 1 x 3 matrix
B) B is 1 x 3 matrix
C) BA is a 3 x 1 matrix
D) (2, 1) entry in AB is - 20
E) (1, 1) entry in B^TA is 17

$\underset{3\times3}{A}\ \underset{3\times1}{B} \rightarrow 3\times1$

$\underset{3\times1}{B}\ \underset{3\times3}{A} \rightarrow$ undefined

$AB = [\ 0 \ -17]$

$B^TA = [0 \ -5 \ 6] \begin{bmatrix} 1 & 2 & 0 \\ -1 & 7 & 3 \\ 2 & 4 & -2 \end{bmatrix} = [\ 17 \ \ -11 \ \ 0\ ]$

---

5) Let A be 5 x 7, B be 7 x 4, C be 7 x 4, D be 3 x 4 matrices. Then which one of the following operations is **defined**: (I is 5 x 5 identity matrix)

A) 5C B
B) 3DC
C) $6ABC^T$,
D) 4A + 2B + 3D
E) B + C + I

$6\ \underset{5\times7}{A}\ \underset{7\times4}{B}\ \underset{4\times7}{C^T}$

5×4

$= 5\times7$

1) Let A be 5 x 7, B be 7 x 4, C be 7 x 4, D be 3 x 4 matrices. Then which one of the following operations is **defined**: (I is 5 x 5 identity matrix) 1) ______

A) B + C + I
B) 2A + 2B + 3D
C) 4DC
D) 3C B
E) $5ABC^T$

$5\ A\ B\ C^T$

Same

---

6) Let $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$. Then (3,1) entry in A^{-1} is

A) 0
B) $\frac{1}{2}$
C) A is singular
D) 2
E) -1

11) Let $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$. Then (2,1) entry in A^{-1} is

A) 2
B) 0
C) $\frac{1}{2}$
D) -1
E) A is singular

$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/2 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

---

7) What condition, if any, must a, b, and c satisfy for the following linear system to be **consistent** ?

$$\begin{cases} x_1 + 3x_2 + x_3 = a \\ -x_1 - 2x_2 + x_3 = b \\ 3x_1 + 7x_2 - x_3 = c \end{cases}$$

A) a, b, c are any real numbers.
B) a - 2b - c = 0
C) 2a - 3b + 4c = 0
D) 3a - 2b + c = 0
E) a + 3b - c = 0

12) What condition, if any, must a, b, and c satisfy for the following linear system to be **consistent** ?

$$\begin{cases} x + 3y + z = A \\ -x - 2y + z = B \\ 3x + 7y - z = C \end{cases}$$

A) 3A - 2B + C = 0
B) A - 2B - C = 0
C) 2A - 3B + 4C = 0
D) A, B, C are any real numbers.
E) A + 3B - C = 0

$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & -3a+c \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & 2a+2b + -3a+c \end{bmatrix}$

$2a + 2b - 3a + c = 0$

$2b - a + c = 0$

$-2b + a - c = 0$

---

8) What can we say about the solution to the following system ?

$$\begin{cases} x + y + z = 1 \\ 3x - y + 6z = 7 \\ 6x + 2y + 9z = 10 \end{cases}$$

A) Exactly three solutions
B) Infinitely many solutions
C) Only trivial (zero) solution
D) A unique nonzero solution
E) No solution

3) What can we say about the solution to the following system ? 3

$$\begin{cases} x + y + z = 1 \\ 3x - y + 6z = 7 \\ 5x + y + 8z = 9 \end{cases}$$

A) Only trivial (zero) solution
B) A unique nonzero solution
C) Infinitely many solutions
D) Exactly four solutions
E) No solution

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 6 & 7 \\ 6 & 2 & 9 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 3 & 4 \\ 0 & -4 & 3 & 4 \end{bmatrix} \rightarrow$ x, y Basic
z Free

ignore

---

9) Let A and B be 7 x 7 invertible matrices. Which one is **false**.

A) $(AB)^{-1} = B^{-1}A^{-1}$ ✓
B) $B^0 = I$, where I is 7 x 7 identity matrix ✓
C) $(A^2)^{-1} = (A^{-1})^2$ ✓
D) $(5A)^{-1} = \frac{1}{5}A^{-1}$ ✓
E) The reduced row echelon form of the matrix BA can't be ident matrix.

2) Let A and B be 9 x 9 invertible matrices. Which one is **false**. 2) ______

A) $(A^3)^{-1} = (A^{-1})^3$
B) $(4A)^{-1} = \frac{1}{4}A^{-1}$
C) $A^0 = I$, where I is 9 x 9 identity matrix
D) $(AB)^{-1} = B^{-1}A^{-1}$
E) The reduced row echelon form of the matrix BA can't be identity matrix.

10) For which value(s) of k, the following augmented matrix corresponds to a **consistent** linear system ?

$$\begin{bmatrix} 3 & -4 & k \\ -6 & 8 & 5 \end{bmatrix}$$

A) $k = -\frac{5}{2}$
B) $k = -2$, $k = 5$
C) $k = 0$
D) $k = -10$
E) k is any real number

$$\begin{bmatrix} 1 & -4/3 & k/3 \\ -6 & 8 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4/3 & k/3 \\ 0 & 0 & 2k+5 \end{bmatrix}$$

$2k+5=0 \Rightarrow k=-5/2$

9) For which value(s) of m, the following augmented matrix corresponds to a **consistent** linear system ?

$$\begin{bmatrix} 2 & -3 & m \\ -4 & 6 & 5 \end{bmatrix}$$

A) $m = -10$
B) $m = -\frac{5}{2}$
C) m is any real number
D) $m = -2$, $m = 5$
E) $m = 0$

---

11) What can we say about the solution to the following system ?

$$\begin{cases} 2x - 3y + 7z = 5 \\ 3x + y - 3z = 13 \\ 2x + 19y - 47z = 32 \end{cases}$$

A) No solution
B) Only zero solution
C) Infinitely many solutions
D) One zero solution and one nonzero solution
E) A unique nonzero solution

7) What can we say about the solution to the following system ? 7) ___

$$\begin{cases} 2x - 3y + 7z = 5 \\ 3x + y - 3z = 13 \\ 2x + 19y - 47z = 32 \end{cases}$$

A) Only zero solution
B) A unique nonzero solution
C) Infinitely many solutions
D) No solution
E) One zero solution and one nonzero solution

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 0 & 5.5 & -13.5 & 5.5 \\ 0 & 22 & -54 & 27 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 0 & 1 & -\frac{13.5}{5.5} & 1 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 0 & 1 & -\frac{13.5}{5.5} & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

---

12) Let A be a 5 x 5 matrix and the system $Ax = b$ has a unique solution for each 5 x 1 vector b. Then which one is **false ?** 12) ___

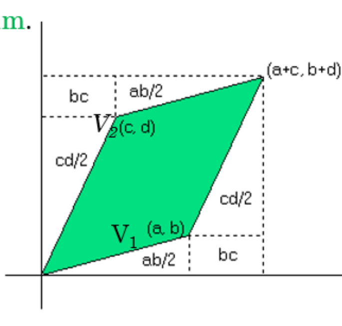
A) The system $Ax = 0$ has infinitely many solutions
B) $Ax = b$ can be solved by elimination method ✓
C) A is invertible ✓
D) The Reduced Row Echelon Form of A is 5 x 5 identity matrix ✓
E) $Ax = b$ can be solved by $x = A^{-1} b$ ✓

4) Let A be a 3 x 3 matrix and the system $Ax = b$ has a unique solution for each 3 x 1 vector b. Then which one is **false ?** 4) ______

A) The system $Ax = 0$ has infinitely many solutions
B) $Ax = b$ can be solved by elimination method
C) A is invertible
D) $Ax = b$ can be solved by $x = A^{-1} b$
E) The Reduced Row Echelon Form of A is 3 x 3 identity matrix

- What is a determinant?
- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix and $V_1 = \begin{pmatrix} a \\ b \end{pmatrix}, V_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, two vectors which coordinates are the first and second rows of the matrix A as in the figure.
- Now calculate the area S of the green parallelogram.

This can be calculated by the surface of the rectangle containing the parallelogram minus the areas of the parts in white. That is



$$S = (a+c)(b+d) - \left(bc + \frac{ab}{2} + \frac{cd}{2} + \frac{ab}{2} + \frac{cd}{2} + bc\right)$$

$= ad - bc$ (cross product)

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$S = ad - bc$

The determinant $\det(A)$ in two dimension is defined to be the surface of the green parallelogram define by the two row vectors of the matrix

$$\det(A) = S = ad - bc$$

Theorem 1

- ***Det(A)=0 ⇔ the matrix is singular*** *(or the corresponding linear system has no unique solution what ever the right hand side vector)*

Or equivalently

- ***Det(A)≠0 ⇔ the matrix is invertibale*** *(or the corresponding linear system has a unique solution what ever the right hand side vector)*

Inverse of a Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

detA.

If $\det(A) \neq 0 \iff A$ is invertible.

THEOREM 2.3.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) *A is invertible.* Ch1
(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.* Ch1
(c) *The reduced row echelon form of A is I_n.* Ch1
([redacted]
(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$. Ch1
(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$. Ch1
(g) $\det(A) \neq 0$. Ch2

---

mcq: Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. Which one is True.

a) $\det A \neq 0$ F ✓
b) $Ax = \begin{bmatrix} 0 \\ 19 \end{bmatrix}$ has a unique solution F ✓
c) A is invertible F ✓
d) $Ax = 0$ has infinitely many solutions True
e) RREF of A is $I_{2\times2}$ F

$\det A = 1 \cdot 8 - 2 \cdot 4 = 0$ ✓

det = 8 − 8 = 0

THEOREM 2.3.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) *A is invertible.* F
(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.* F
(c) *The reduced row echelon form of A is I_n.* F
([redacted]
(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$. F
(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$. F
(g) $\det(A) \neq 0$. False

Definition

Let A be a $n \times n$ matrix

- The (i,j)-minor of A, denoted M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column from A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots a_{2j} \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nj} \cdots & a_{nn} \end{bmatrix} \Rightarrow M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots a_{2j} \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nj} \cdots & a_{nn} \end{vmatrix}$$

- The (i,j)-cofactor of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$

example. Let $A = \begin{bmatrix} 1 & 2 & -7 \\ 1 & 0 & 3 \\ 0 & 8 & 5 \end{bmatrix}$

$M_{11} = -24$ $\quad M_{12} = 5$ $\quad M_{13} = 8$

$C_{11} = (-1)^{1+1} \cdot -24 = -24$ $\quad C_{12} = (-1)^{1+2} \cdot 5 = -5$ $\quad C_{13} = (-1)^4 \cdot 8 = 8$

$M_{21} = 66$ $\quad M_{22} = 5$ $\quad M_{23} = 8$

$C_{21} = (-1)^3 \cdot 66 = -66$ $\quad C_{22} = (-1)^4 \cdot 5 = 5$ $\quad C_{23} = (-1)^5 8 = -8$

$M_{31} = 6$ $\quad M_{32} = 10$ $\quad M_{33} = -2$

$C_{31} = (-1)^4 \cdot 6 = 6$ $\quad C_{32} = (-1)^5 \cdot 10 = -10$ $\quad C_{33} = (-1)^6 \cdot -2 = -2$.

mcq: Let A be a 1955×1955 matrix.

$M_{29} + C_{29}$ is ($C_{29} = -M_{29}$)

a) 0 (circled)
b) −1
c) 129
d) 92
e) 1955

$C_{29} = (-1)^{11} \cdot M_{29}$

$C_{29} = -M_{29}$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion along the j^{th} column)

(fix j, i = 1, n)

d

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion along the i^{th} row)

(fix i, j = 1, n)

---

example. Let $A = \begin{bmatrix} 1 & 2 & -7 \\ 1 & 0 & 3 \\ 0 & 8 & 5 \end{bmatrix}$

a) cofactor expansion along row 1: (i = 1)

det $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
$= 1 \cdot -24 + 2 \cdot -5 - 7 \cdot 8$
$= -24 + -10 + -56 = -90$.

b) Cofactor expansion along row 2 (i = 2)

$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$
$= 1 \cdot -66 + 0 \cdot 5 + 3 \cdot -8$
$= -66 + -24 = -90$.

c) Cofactor expansion along row 3 (i = 3)

$a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$
$= 0 \cdot 6 + 8 \cdot -10 + 5 \cdot -2$
$= -90$.

d) Cofactor expansion along column 1 $(J=1)$

$a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= 1 \cdot -24 + 1 \cdot -66 + 0 \cdot ?$

$= -90.$

e) Cofactor expansion along column 2 $(J=2)$

$a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$2 \cdot -5 + 0 \cdot ? + 8 \cdot -10 = -90$

f) Cofactor expansion along column 3 $(J=3)$

$a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

$-7 \cdot 8 + 3 \cdot -8 + 5 \cdot -2$

$= -56 + -24 + -10 = -90.$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$ det(A)?

$C_{11} = (-1)^2 \cdot -4 = -4$

$C_{12} = (-1)^3 \cdot -11 = 11$

row 1: $(i=1)$ $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$3 \cdot -4 + 1 \cdot 11 + 0 \cdot ?$

$= -12 + 11 = -1.$

Remark: A is invertible because $\det A \neq 0$.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$ which one is False

✓ a) $\det A = -1$

✓ b) A is invertible

✓ c) RREF is $I_{3\times 3}$

✓ d) $Ax = 0$ has only trivial (zero) solution

(e) $Ax = b$ may have no solution, (3×3, 3×1) b is 3×1 vector.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

det column 2 $(J=2)$ $a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42}$

$0 \cdot + 1 \cdot C_{22} + 0 \cdot + 0.$

$\det = C_{22} = (-1)^2 \cdot M_{22} = \det\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right)$ B

Column 2 $(J=2)$ $b_{12}C_{12} + b_{22}C_{22} + b_{32}C_{32}$

$= -2 \cdot (-1)^{2+2} \cdot 3$

$= -6.$

---

23. $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$ Cofactor expansion along column 1

$a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= 1 \cdot C_{11} + 1 \cdot C_{21} + 1 \cdot C_{31}$

$= 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 = 0$

$C_{11} = (-1)^2 \cdot 0 = 0$

$C_{21} = (-1)^3 \cdot 0 = 0$

$C_{31} = (-1)^4 \cdot 0 = 0$

33. In each part, show that the value of the determinant is independent of θ.

(a) $\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ and calculate A^{-1}

$\det = \sin^2\theta + \cos^2\theta = 1.$

A is invertible: $A^{-1} = \frac{1}{1}\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the *matrix of cofactors from A*.

The transpose of the *matrix of cofactors from A* is called the *adjoint of A* and is denoted by *adj(A)*

$$adj(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- **Theorem** (Inverse of a Matrix using its Adjoint)

  If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

A is invertible $\Longleftrightarrow$ $\det A \neq 0$

ex. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$C_{11} = d$ $\quad C_{21} = -b$

$C_{12} = -c$ $\quad C_{22} = a$ $\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & c \\ -b & a \end{bmatrix}^T$

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$C_{11} = 40$ $\quad C_{21} = -16$ $\quad C_{31} = -9$

$C_{12} = -13$ $\quad C_{22} = 5$ $\quad C_{32} = -(-3) = 3$

$C_{13} = -5$ $\quad C_{23} = -(-2) = 2$ $\quad C_{33} = 1$

det row3 $(i=3)$: $a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$= 1 \cdot C_{31} + 0 \cdot C_{32} + 8 \cdot C_{33}$

$= -9 + 8 = -1$

$A^{-1} = \frac{1}{-1}\begin{bmatrix} 40 & -13 & -5 \\ -16 & 5 & 2 \\ -9 & 3 & 1 \end{bmatrix}^T$

$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

a) is A invertible

b) Find A^{-1}.

row-operations $\quad$ $\frac{1}{\det A} \cdot adj(A)$

a) row3: $(i=3)$

$a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$C_{31} = 12$

$C_{32} = -10$

$2 \cdot 12 + -4 \cdot -10 + 0 \cdot C_{33} = 64$

A is invertible because $\det A \neq 0$.

Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$. which one is False.

a) A is a square matrix ✓

b) A is invertible ✓

✓c) $Ax = b$ has a unique solution, where b is a 3x1 vector

(d) $\det A = 77$

✓e) $C_{11} = 12$

- A determinant of an $n \times n$ diagonal matrix D is given by

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\det(D) = d_1 d_2 \ldots d_n$$

ex. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$. which one is False.

a) A is invertible ✓

b) A is diagonal ✓

c) det = −49 ✓

(e) $Ax = b$ has infinitely many solutions

det = 1·7·−7 = −49

ex. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det = 0$

$\Rightarrow A^{-1}$ does not exist.

- Determinant of a triangular matrix

  If A is an $n \times n$ triangular matrix (upper triangular, lower triangular,)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Then det(A) is the product of the entries of the main diagonal

$$\det(A) = a_{11}a_{22} \ldots a_{nn}$$

mcq: Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & -3 & 0 & 0 \\ 3 & 1 & 4 & 0 \\ 2 & -7 & 10 & 0 \end{bmatrix}$. which one is True

F a) A is upper triangular .

F b) det = −24 $\quad$ det = 0

T (c) $Ax = 0$ has infinitely many solutions

F d) A is invertible

F e) A is diagonal

$$A=\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = ?$$

det = 2.-3.6.9.4
= .....
A is invertible.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

det = 1.2.5 = 10 — A^{-1} exist

det = 0 — A^{-1} doesnot exist

Let A be a square matrix

If A has a row of zeros or a column of zeros, then

$$\det(A) = 0.$$

Theorem 6

Let A be a square matrix

$$\det(A) = \det(A^T)$$

1. If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then

$$\det(B) = k \det(A)$$

2. If B is the matrix that results when two rows or two columns of A are interchanged, then

$$\det(B) = - \det(A)$$

3. If B results from adding to any row of A a multiple of any other row **or** by adding to any column of A a multiple of any other column, then

$$\det(B) = \det(A)$$

Example:

mcq: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{21} & a_{22} \\ 3a_{11} & 3a_{12} \end{bmatrix}$

If det A = -7, det B is

(a) 21 b) 14 c) 7 d) 0 e) 77

$$A: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{-7 \to 7} \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \xrightarrow{7 \to 21} \begin{bmatrix} a_{21} & a_{22} \\ 3a_{11} & 3a_{12} \end{bmatrix} = B$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{11} & a_{22}+2a_{12} & a_{23}+2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

det A = 5, det B is

(a) 5 b) 10 c) 15 d) 0 e) 1907

$$A \xrightarrow{R_2 \to 2R_1 + R_2} B$$

Theorem 7

If A is a square matrix with two proportional rows or two proportional column, then

$$\det(A) = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 10 & 19 \end{bmatrix}$$

$$A: \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad B: \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

C2 = -2.C1
det A = 0
A^{-1} does not exist

R4 = -3.R1
det B = 0
B^{-1} does not exist.

Monday: There is a class
Tuesday: m1-exam
Wednesday: I solve exam questions

- Determinant of **det(kA)**
- If

$$A=\begin{bmatrix} ka_{11} & ka_{12} & \cdots ka_{1j}\cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots ka_{2j}\cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ ka_{n1} & ka_{n2} & \cdots ka_{nj}\cdots & ka_{nn} \end{bmatrix}$$

- Apply propriety 1. to each row

$$\det(A)=k^n\begin{vmatrix} a_{11} & a_{12} & \cdots a_{1j}\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots a_{2j}\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nj}\cdots & a_{nn} \end{vmatrix}$$

$\det(kA) = (k)^n\det(A)$

A is n×n

If A and B are square matrices of the same size, then

$\det(AB) = \det(A)\det(B)$

$\det(ABC)=\det(A)\cdot\det(B)\cdot\det(C)$

$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ $\det(AB)=\det(A)\cdot\det(B)$

$B=\begin{bmatrix}0 & 1\\ 3 & 6\end{bmatrix}$ $6 = -2 \cdot -3$

$AB=\begin{bmatrix}6 & 13\\ 12 & 27\end{bmatrix}$ $6 = 6$

$\det(A+B) \neq \det(A)+\det(B)$

: $A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}, B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}, A+B=\begin{bmatrix}4 & 3\\ 3 & 8\end{bmatrix}$

$\det(A+B) \neq \det(A)+\det(B)$

$23 \neq 1 + 8$

$$\det(A^{-1})=\frac{1}{\det(A)}$$

$\det(A\cdot A^{-1})=\det(I)$

$\det(A)\cdot\det(A^{-1}) = 1$

$\det(A^{-1}) = \frac{1}{\det A}$.

A triangular (diagonal) matrix is invertible if and only if its diagonal entries are all nonzero

If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

$\det(A) = \det(A^T)$

If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, than $\det(B) = k\det(A)$

If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$

If A has two proportional rows or two proportional columns, then $\det(A) = 0$

$\det(kA) = (k)^n\det(A)$

$\det(AB) = \det(A)\det(B)$ ten $\det(A^{-1})=\frac{1}{\det(A)}$

$\det(A+B) \neq \det(A)+\det(B)$

mcq. Let A be a 3x3 matrix and det A = 2. Which one is false.

a) A is invertible ✓

✓ b) $\det(A^{-1}) = \frac{1}{2}$. $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{2}$ ✓

✓ c) $\det(B^T) = 2$ ✓

False. (d) $\det(4A) = 8$. $\det(4A) = 4^3 \cdot 2 = 128$

✓ e) $Ax=b$ must have a unique solution, where b is a 3x1 matrix.

$\det(4A) = 4^3 \cdot 2 = 128$

---

mcq: Let A and B be 5x5 matrices, $\det(A)=2$, $\det(B)=-1$. Then $\det(2A^TB^{-1})$ is

a) 16 (b) -64 c) 32 d) 1 e) 1855

$\det(2\underbrace{A^T\cdot B^{-1}}_{5\times5}) = 2^5 \cdot \det(A^T)\cdot\underbrace{\det(B^{-1})}_{\frac{1}{\det B}}$

$= 32 \cdot 2 \cdot \frac{1}{-1}$

$= -64$.

---

mcq. Let A, B, C, D, E be 2x2 matrices. and $\det A=-1$, $\det B=3$, $\det(C^{-1})=-2$, $\det(D^T)=4$, $\det(E)=1$. Then $\det(2A^{-1}B^{-1}C^TD^{-1}E^T)$ is

a) $\frac{2}{3}$ b) $\frac{3}{2}$ (c) $\frac{1}{6}$ f) 0 e) 1907

$\det(C^{-1}) = -2$

$\det(C) = -\frac{1}{2}$

$\det(2\underbrace{A^{-1}B^{-1}C^TD^{-1}E^T}_{2\times2}) =$

$= 2^2 \cdot \det(A^{-1})\cdot\det(B^{-1})\cdot\det(C^T)\cdot\det(D^{-1})\cdot\det(E^T)$

$= 4 \cdot \frac{1}{-1} \cdot \frac{1}{3} \cdot -\frac{1}{2} \cdot \frac{1}{4} \cdot 1$

$= \frac{1}{6}$

---

mcq. Let A, B, C be 5x5 matrices, $\det(A)=-1$, $\det(B^{-1})=3$ and $\det(2A^TBC^T)=32$ then $\det(C^{-1})$ is

✓ (a) $-\frac{1}{3}$ b) 3 c) 1 d) 0 f) 5

$\det(2\underbrace{A^TBC^T}_{5\times5}) = 32$

$2^5 \cdot -1 \cdot \frac{1}{3} \cdot \det(C^T) = 32$

$-\frac{1}{3}\det(C^T) = 1$

$\det(C^T) = -3$

⇓

$\det(C) = -3$

⇓

$\det(C^{-1}) = \frac{1}{\det(C)} = \frac{1}{-3} = -\frac{1}{3}$

Let A be a 7x7 matrix and $A^T = -A$. Det(A) is

a) 7 b) -1 (c) 0 d) 77 e) 49

$\det(A^T) = \det(-1\cdot\underbrace{A}_{7\times7})$

$\det(A) = (-1)^7\cdot\det(A)$

$\det(A) = -\det(A)$ ($x=-x$, $2x=0$, $x=0$)

$2\det(A) = 0$

$\det(A) = 0$

ex.

Let A be a 10x10 matrix and $A^T = -A$. det A is

a) 0
b) 1
c) -1
d) 10
e) can't be calculated (circled)

$\det(A^T) = \det(-A)$

$\det A = (-1)^{10} \det(A)$

$\det A = \det A$.

Tuesday: exam (6-7:30)

Wednesday: Solve exam.

Next week: 2.2

2.1
2.3
2.2

$16 \times 1.5 = 24$. max 22.

- Determinant of det(*kA*)
- If

$$A=\begin{bmatrix} ka_{11} & ka_{12} & \cdots ka_{1j}\cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots ka_{2j}\cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ ka_{n1} & ka_{n2} & \cdots ka_{nj}\cdots & ka_{nn} \end{bmatrix}$$

- Apply propriety 1. to each row

$$\det(A)=k^n\begin{vmatrix} a_{11} & a_{12} & \cdots a_{1j}\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots a_{2j}\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nj}\cdots & a_{nn} \end{vmatrix}$$

det(*kA*) = (*k*)^{n}det(*A*)

A is n×n

If *A* and *B* are square matrices of the same size, then

det(*AB*) = det(*A*) det(*B*)

det(ABC) = det(A)·det(B)·det(C)

$A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ det(AB) = det(A)·det(B)

$B=\begin{bmatrix}0&1\\3&6\end{bmatrix}$ 6 = −2 · −3

$AB=\begin{bmatrix}6&13\\12&27\end{bmatrix}$ 6 = 6

---

det(A+B) ≠ det(A)+det(B)

: $A=\begin{bmatrix}1&2\\2&5\end{bmatrix}, B=\begin{bmatrix}3&1\\1&3\end{bmatrix}, A+B=\begin{bmatrix}4&3\\3&8\end{bmatrix}$

det(A+B) ≠ det(A)+det(B)

23 ≠ 1 + 8

$$\det(A^{-1})=\frac{1}{\det(A)}$$

$\det(A\cdot A^{-1})=\det(I)$

$\det(A)\cdot\det(A^{-1}) = 1$

$\det(A^{-1}) = \frac{1}{\det A}$.

A triangular (diagonal) matrix is invertible if and only if its diagonal entries are all nonzero

If *A* has a row of zeros or a column of zeros, then det(*A*) = 0.

det(*A*) = det(A^T)

If *B* is the matrix that results when a single row or single column of *A* is multiplied by a scalar *k*, than det(*B*) = *k* det(*A*)

If *B* is the matrix that results when two rows or two columns of *A* are interchanged, then det(*B*) = - det(*A*)

If *A* has two proportional rows or two proportional columns, then det(*A*) = 0

det(*kA*) = (*k*)^{n}det(*A*)

det(*AB*) = det(*A*) det(*B*) ten $\det(A^{-1})=\frac{1}{\det(A)}$

det(A+B) ≠ det(A)+det(B)

mcq. Let A be a 3×3 matrix and det A = 2. Which one is false.

a) A is invertible ✓

✓ b) $\det(A^{-1})=\frac{1}{2}$. $\det(A^{-1})=\frac{1}{\det A}=\frac{1}{2}$ ✓

✓ c) $\det(A^T)=2$ ✓

False. d) det(4A) = 8. $\det(4A)=4^3\cdot 2=128$

✓ e) Ax=b must have a unique solution, where b is a 3×1 matrix.

$\det(4A)=4^3\cdot 2=128$

---

mcq: Let A and B be 5×5 matrices, det(A) = 2, det(B) = −1. Then $\det(2A^TB^{-1})$ is

a) 16 (b) −64 c) 32 d) 1 e) 1855

$\det(2\underbrace{A^T B^{-1}}_{5\times5}) = 2^5\cdot\det(A^T)\cdot\det(B^{-1})$, $\det(B^{-1})=\frac{1}{\det B}$, $\det B=-1$

$= 32\cdot 2\cdot$

$= -64$.

---

mcq. Let A, B, C, D, E be 2×2 matrices and det A = −1, det B = 3, $\det(C^{-1})=-2$, $\det(D^T)=4$, det(E) = 1. Then $\det(2A^{-1}B^{-1}C^TD^{-1}E^T)$ is

a) $\frac{2}{3}$ b) $\frac{3}{2}$ (c) $\frac{1}{6}$ d) 0 e) 1907

$\det(C^{-1})=-2$

$\det(C)=-\frac{1}{2}$

$\det(2\underbrace{A^{-1}B^{-1}C^TD^{-1}E^T}_{2\times2}) =$

$= 2^2\cdot\det(A^{-1})\cdot\det(B^{-1})\cdot\det(C^T)\cdot\det(D^{-1})\cdot\det(E^T)$

$= 4\cdot\frac{1}{-1}\cdot\frac{1}{3}\cdot -\frac{1}{2}\cdot\frac{1}{4}\cdot 1$

$= \frac{1}{6}$

---

mcq. Let A, B, C be 5×5 matrices, det(A) = −1, $\det(B^{-1})=3$ and $\det(2A^TBC^T)=32$ then $\det(C^{-1})$ is

✓ (a) $-\frac{1}{3}$ b) 3 c) 1 d) 0 f) 5

$\det(2\underbrace{A^TBC^T}_{5\times5}) = 32$

$2^5\cdot -1\cdot\frac{1}{3}\cdot\det(C^T) = 32$

$-\frac{1}{3}\det(C^T) = 1$

$\det(C^T) = -3$

⇓

$\det(C) = -3$

⇓

$\det(C^{-1}) = \frac{1}{\det(C)} = \frac{1}{-3} = -\frac{1}{3}$

Let A be a 7×7 matrix and $A^T=-A$. Det(A) is

a) 7 b) −1 (c) 0 d) 77 e) 49

$\det(A^T) = \det(-1\cdot\underbrace{A}_{7\times7})$

$\det(A) = (-1)^7\cdot\det(A)$

$\det(A) = -\det(A)$ $\left(\begin{array}{l} x=-x \\ 2x=0 \\ x=0 \end{array}\right)$

$2\det(A)=0$

$\det(A)=0$

ex.

Let A be a 10×10 matrix and $A^T = -A$. $\det A$ is

a) 0
b) 1
c) -1
d) 10
(e) can't be calculated

$\det(A^T) = \det(-A)$

$\det A = (-1)^{10} \det(A)$

$\det A = \det A$.

Tuesday: exam (6 - 7.30)

Wednesday: Solve exam.

Next week: 2.2

2.1
2.3
2.2

$16 \times 1.5 = 24$. max 22.

Evaluating Determinants by Row Reduction

- The idea is to reduce the matrix into an upper-triangular using the Gauss-Elimination algorithm, but by taking into account the proprieties 1, 2,and 3 (see slides 32-35). That is,

1. To get the leader 1, we need to multiply the determinant by the nonzero entry after dividing the row by the same value.
2. If any two rows are interchanged, the determinant must be multiplied by -1
3. For the rows operation involving the sum of any row with a multiplier of any other row, the determinant don't change.

   Remark: don't multiply the considered row by any coefficient, only the one with a leader 1 could be multiplied.

$|A|$ = determinant

$$\begin{vmatrix} 4 & 2 \\ 5 & 7 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} \checkmark$$

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

a) Cofactor expansion

b) Row-Reduction.

a) Row 1: $\cancel{a_{11}C_{11}} + a_{12}C_{12} + a_{13}C_{13}$ $\quad C_{12} = -(-15)$

$= 1 \cdot 15 + 5 \cdot 30 = 165.$ $\quad C_{13} = 30$

b) $$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -1 \cdot 3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -1 \cdot 3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$= -1 \cdot 3 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = -1 \cdot 3 \cdot 1 \cdot 1 \cdot -55 = 165$$

$$A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

a) Cofactor expansion

b) Row-Reduction.

a) Row 2: $\underbrace{a_{21}C_{21}}_{-1 \cdot 5} + \underbrace{a_{22}C_{22}}_{-1 \cdot -4} + \cancel{\underbrace{a_{23}C_{23}}_{0}}$

$C_{21} = -(-5)$ $\quad = -5 + 4 = -1.$

$C_{22} = -4$

b) $$\begin{vmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{vmatrix} = - \begin{vmatrix} -1 & -1 & 0 \\ 2 & 5 & 5 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= -1 \cdot -1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 5 \\ 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 5 \\ 0 & 2 & 3 \end{vmatrix} \quad -\frac{10}{3}+3$$

$$= 3 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 5/3 \\ 0 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 5/3 \\ 0 & 0 & -\frac{1}{3} \end{vmatrix}$$

$$= \cancel{3} \cdot 1 \cdot 1 \cdot -\frac{1}{\cancel{3}} = -1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}.$$ which one is True

a) it is upper triangular X

b) it is diagonal X

X c) $Ax=0$ has infinitely many solution

X d) det = -1907

X e) A is singular

F) $Ax=b$ must have a unique solution

$$A = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 1 & -26 \end{vmatrix}$$

$$= 7 \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 1 & -26 \end{vmatrix} = 7 \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & -26 \end{vmatrix}$$

$$= 7 \cdot -1 \cdot \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -26 \\ 0 & 0 & 3 & 0 \end{vmatrix} = 7 \cdot -1 \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -26 \\ 0 & 0 & 0 & 78 \end{vmatrix}$$

$= 7 \cdot -1 \cdot 1 \cdot 1 \cdot 1 \cdot 78$ $\quad 78 \times 7 = 546$

$= -546$

Linear System ($Ax=b$)

elimanition

$x = A^{-1}b$ ($\det A \neq 0$)

Cramer's rule ✓ ($\det A \neq 0$)

1. Cramer's Rule - two equations

If we are given a pair of simultaneous equations

$$\begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

then x, and y can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\begin{aligned} 3x + 4y &= -14 \\ -2x - 3y &= 11 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -14 \\ 11 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2$$

$$y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5.$$

mcq: We are given $\begin{cases} 3x + 4y = -14 \\ -2x - 3y = 11 \end{cases}$. which one is True

a) it has infinitely many solutions. X

b) it can't be solved by Cramer's rule X

c) if you put it into $Ax=b$, $\det A = 0$ X

d) $\begin{cases} x = 5 \\ y = -15 \end{cases}$ is a solution X

e) it has a unique solution ✓

F) it can be solved by elimanition ✓

b)

$$\begin{aligned} x + 2y + 3z &= 17 \\ 3x + 2y + z &= 11 \\ x - 5y + z &= -5 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}}$$

(exercise)

b) $x = 1$, $y = 2$, $z = 4$

78/100. ✓

Quiz 2: Ch2 + Equivalent statements

Definitions

- The eigenvalues of an $n \times n$ matrix A are the number λ for which there is a nonzero $v \neq 0$ with
$$Av = \lambda v.$$

- The eigenvectors of A are the nonzero vectors $v \neq 0$ for which there is a number λ with
$$Av = \lambda v.$$

- If $Av = \lambda v$ for $v \neq 0$, then

v is an eigenvector associated with the eigenvalue λ, and vice versa.

$$Av = \lambda v$$
$$Av - \lambda v = 0$$
$$(A - \lambda I)v = 0$$
$$(A - \lambda I)v = 0, \quad v \neq 0$$
$$\boxed{\det(A - \lambda I) = 0}$$

$$\det(A - \lambda I) = 0$$

This is called characteristic equation of A

And $\det(A - \lambda I)$ is called the characteristic polynomial

1. Calculate $\det(A - \lambda I) = 0$, and solve for λ. Note that $\det(A - \lambda I)$ is a polynomial of degree n

2. For each eigenvalue λ found in step 1, solve the linear system
$$(A - \lambda I)X = 0,$$
to find the eigenvectors X

$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

a) Calculate $A - \lambda I_{2\times 2}$

b) Calculate $\det(A - \lambda I)$ (Characteristic polynomial)

c) Solve $\det(A - \lambda I) = 0$ (Find eigenvalues)

d) Find eigenvectors

a) $A - \lambda I = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$

Substract λ from diogonal entries.

b) $\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 12$

$= 2 - \lambda - 2\lambda + \lambda^2 - 12$

Characterstic polynomial $\boxed{= \lambda^2 - 3\lambda - 10}$

c) $\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda_1 = 5 \quad \lambda_2 = -2.$

d) Eigenvectors

$\lambda = 5$

we solve $(A - \lambda I)V = 0$

$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

~~$-4v_1 + 3v_2 = 0$~~

$\begin{bmatrix} -4 & 3 & 0 \\ 4 & -3 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -3/4 & 0 \\ 4 & -3 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_1 - \frac{3}{4} v_2 = 0$

v_2 is free

$v_1 = \frac{3}{4} v_2$, v_2 free

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

$v_2 \neq 0$

eigenvectors: $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \ldots$

$\lambda = -2$

$(A - \lambda I)v = 0$

$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$3v_1 + 3v_2 = 0$

$4v_1 + 4v_2 = 0$

$v_1 + v_2 = 0$

$v_1 = -v_2$, v_2 free

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$v_2 \neq 0$

eigenvectors: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3/4 \\ -3/4 \end{bmatrix} \ldots$

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$. Eigenvalues are

a) 5, −2 b) 1, 2 c) 2, −3 d) 0, 2 e) 0, 5

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$. By knowing that $\lambda = -2$ is an eigenvalue, an eigenvector is

a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ $= 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

e) $\begin{bmatrix} 2/3 \\ 3/4 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 8 = 0$

$-3 - 3\lambda + \lambda + \lambda^2 + 8 = 0$

$a = 1$, $b = -2$, $c = 5$

$1 \cdot \lambda^2 - 2\lambda + 5 = 0$

$\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda_{1,2} = \dfrac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2}$ $= \sqrt{-16} = \sqrt{16 \cdot i^2} = 4i$

$\lambda_{1,2} = \dfrac{2 \pm 4i}{2}$

$\lambda_{1,2} = 1 \pm 2i$

$\boxed{\lambda_1 = 1 + 2i \quad \lambda_2 = 1 - 2i}$

EigenVectors

$\lambda = 1+2i$

$(A-\lambda I)v = 0$

$$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$(2-2i)v_1 - 2v_2 = 0$ $\times(1+i)$

$4v_1 + (-2-2i)v_2 = 0$

$4v_1 - 2(1+i)v_2 = 0$

$2(1-i)v_1 - 2v_2 = 0$

$2(1-i)v_1 = 2v_2$

$v_1 = \frac{1}{1-i}v_2$

v_2 free

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{1-i}v_2 \\ v_2 \cdot 1 \end{bmatrix} = v_2\begin{bmatrix} \frac{1}{1-i} \\ 1 \end{bmatrix}$$

$\frac{1}{1-i} \cdot (1+i) = \frac{1+i}{1-i^2} = \frac{1+i}{2}$, $i^2 = -1$

$$v_2\begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$$

i is the complex number such that $i^2 = -1$.

$\lambda = 1-2i$

eigenvectors

$$v_2 \cdot \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}.$$

Complex numbers:

$2+3i \xrightarrow{\text{conjugate}} 2-3i$

$(1+i)(2-3i) = 2-3i+2i-3i^2 = 2-i+3 = 5-i$ ($i^2 = -1$)

Rationalize

$$\frac{2(4-2i)}{(4+2i)(4-2i)} = \frac{8-4i}{16-(4i^2)} = \frac{8-4i}{20}$$ ($i^2 = -1$)

$(a+b)(a-b)$

$a^2 - b^2$

$a = 4$ $b = 2i$

$4^2 - (2i)^2$

$16 - 4(i^2)$ ($i^2 = -1$)

20

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Eigenvalues, eigenvectors.

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$\det(A-\lambda I) \overset{\text{column 2}}{=} a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= (2-\lambda) \cdot [-\lambda(3-\lambda)+2]$

$= (2-\lambda)[\lambda^2 - 3\lambda + 2] = 0$

$= (2-\lambda) \cdot (\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 2$ $\lambda_2 = 2$ $\lambda_3 = 1$

Eigenvalues: 2, 2, 1

Definition: Algebraic Multiplicity

When solving the equation $\det(A-\lambda I) = 0$, we looking for the roots of the characteristic polynomial $\det(A-\lambda I)$.

An algebraic multiplicity of an eigenvalue λ is the number of times λ is repeated as a root of the characteristic polynomial.

Eigenvalues: 2, 1

2: arithmetic multiplicity = 2

1: arithmetic multiplicity = 1

Next time: Eigenvectors.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Eigenvalues, eigenvectors.

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$\det(A - \lambda I) \overset{\text{column 2}}{=} a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= (2-\lambda)\cdot\left[-\lambda(3-\lambda)+2\right]$

$= (2-\lambda)\left[\lambda^2 - 3\lambda + 2\right] = 0$

$= (2-\lambda)\cdot(\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 2 \quad \lambda_2 = 2 \quad \lambda_3 = 1$

Eigenvalues: 2, 2, 1

Eigenvectors

$\lambda = 2$: $(A - \lambda I)v = 0$ for $v (\neq 0)$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \rightarrow \begin{cases} -2v_1 - 2v_3 = 0 \\ v_1 + v_3 = 0 \\ v_1 + v_3 = 0 \end{cases}$$

3x3, 3x1

$$\begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$v_1 + v_3 = 0$

v_2 is free

v_3 free

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_3 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \cdot 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -v_3 \cdot 1 \\ 0 \\ v_3 \cdot 1 \end{bmatrix} = v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors: $\left\{ v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ v_2, v_3 \neq 0 \right\}$

ex. which one is an eigenvector

a) $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}$

d) $\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ ~~e)~~ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 1$ $(A - \lambda I)v = 0$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$v_1 + 2v_3 = 0 \rightarrow v_1 = -2v_3$

$v_2 - v_3 = 0 \rightarrow v_2 = v_3$

v_3 free

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2v_3 \\ v_3 \\ v_3 \end{bmatrix} = v_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \ v_3 \neq 0$$

mcq: which is an eigenvectors.

~~a)~~ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)\cdot(-2-\lambda)\cdot(3-\lambda) = 0$

$\Rightarrow \lambda = 1 \quad \lambda = -2 \quad \lambda = 3$

> If A is triangular (upper, lower, diagonal) then eigenvalues are the diagonal entries

mcq: which one is an eigenvector for $\lambda = 1$:

~~a)~~ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$(A - \lambda I)v = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0v_1 + 1v_2 + 0v_3 = 0$

$0v_1 + 0v_2 + 1v_3 = 0$

$v_2 = 0$

$v_3 = 0$

v_1 free

eigenvectors: $\begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(Eigenvectors: Exercise) ✓

$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$ $C_{21} = (-1)^3(-1(1-\lambda)) = 1 - \lambda$

$\det \overset{\text{column 1}}{=} a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$(1-\lambda)\left((2-\lambda)(1-\lambda)-1\right) - 1\cdot(1-\lambda) = 0$

$(1-\lambda)\left[(2-\lambda)(1-\lambda) - 1 - 1\right] = 0$

$2 - 2\lambda - \lambda + \lambda^2 - 2$

$(1-\lambda)(\lambda^2 - 3\lambda) = 0$

$\lambda(\lambda - 3)$

$\lambda = 1 \quad \lambda = 0 \quad \lambda = 3$ (eigenvectors: exercise)

The product of the n eigenvalues equals the determinant.
The sum of the n eigenvalues equals the sum of the n diagonal entries.

trace

um of the entries along the main diagonal is called the *trace* of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textit{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \quad (6)$$

$\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = \det$

1. Find the sum & product of the Eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Solution:

$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace} = -1 = 2 + 3 + -6$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 2\cdot -19 + 1\cdot -1\cdot -8 + 2\cdot -5$

$= -38 + 8 + -10 = -40$

Let A be this matrix. which one is false.

a) A is invertible ✓

b) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -40$ ✓

c) $\lambda_1 + \lambda_2 + \lambda_3 = -1$ ✓ ✓

d) $Ax = 0$ has only zero solution

(e) RREF is not I

example. Let A be a 3x3 matrix, and eigenvalues are 0, 1, −1. Which one is true.

a) A is invertible ✗ c) trace = 1 ✗ $= 0 + 1 + -1 = 0$

b) det = −1 ✗ $= 0 \cdot -1 \cdot 1 = 0$

(d) $Ax = 0$ has infinitely many solutions

Let A be an n×n matrix.

$$\text{Trace} = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$$\det = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$$

---

2. The product of two Eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **is 16. Find the third**

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A)$

$16 \cdot \lambda_3 = \det(A)$

$C_{11} = 9 - 1 = 8$

$C_{12} = -[-4] = 4$

$C_{13} = -4$

$\det \overset{\text{row 1}}{=} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$6 \cdot 8 + -2 \cdot 4 + 2 \cdot -4$

$48 - 8 - 8 = 32$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det = 32$

$16 \cdot \lambda_3 = 32 \Rightarrow \lambda_3 = 2$ ✓

which one is True for A:

a) det = 0 F

b) $\lambda = 0$ is an eigenvalue F

(c) A is invertible T

F d) Ax = b must have infinitely many

F e) Ax = 0 must have infinitely many

---

3. Two Eigenvalues of the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **are 3 and 0. What is the third**

trace = 18

* $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace} = 18$

$3 + 0 + \lambda_3 = 18 \Rightarrow \lambda_3 = 15$ ✓

(Shortcut)

* $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det$ Fails

$3 \cdot 0 \cdot \lambda_3 = 0$

$0 \cdot \lambda_3 = 0 \Rightarrow \lambda_3 = ?$

* $A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$

$\det = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= (8-\lambda) \cdot [(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8]$

$+ 2[24 - (2(7-\lambda))] = 0$

Solve

$\lambda_1 = 3$

$\lambda_2 = 0$

$\lambda_3 = 15$

---

trace =

4. If 3 and 15 are two Eigenvalues of the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **then find the third eigenvalue and hence = |A|**

* $\text{trace} = \lambda_1 + \lambda_2 + \lambda_3 = 18$

$3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$

* $\det = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$\det = 3 \cdot 15 \cdot \lambda_3$

$0 = 3 \cdot 15 \cdot \lambda_3 \Rightarrow \lambda_3 = 0$

$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

$\det = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 8 \cdot 5 + -6 \cdot 10 + 2 \cdot 10$

$= 0$

$C_{11} = 5$

$C_{12} = -(-10) = 10$

$C_{13} = 10$

If A is an n × n matrix, then the following statements are equ

(a) *A is invertible.* ✓ Ch1

(b) **Ax = 0** *has only the trivial solution.* Ch1

(c) *The reduced row echelon form of A is* I_n. Ch1

(d) ~~A is expressible as a product of elementary matrices.~~

(e) **Ax = b** *is consistent for every n × 1 matrix* **b**. Ch1

(f) **Ax = b** *has exactly one solution for every n × 1 matrix* Ch1

(g) $\det(A) \neq 0$. Ch2

0 is not an eigenvalue Ch5

−2, −2, −2, −1, 17

example. Let A be a 5x5 matrix and $\lambda_1 = \lambda_2 = \lambda_3 = -2$, $\lambda_4 = -1$, trace = 10.

Determine whether the following statements are True

$\text{trace} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$

$10 = -2 + -2 + -2 + -1 + \lambda_5$

$10 = -7 + \lambda_3$

$\lambda_5 = 17$

- $\lambda_5 = 17$ True
- F • det = 0 $\det = -2 \cdot -2 \cdot -2 \cdot -1 \cdot 17 = 136$
- F • A is not invertible
- F • Ax = b must have no solution
- T • Ax = 0 must have only zero solution
- T • RREF of A is $I_{5\times5}$
- T • Arithmetic multiplicity of $\lambda = -2$ is 3.

---

next week: Ch3

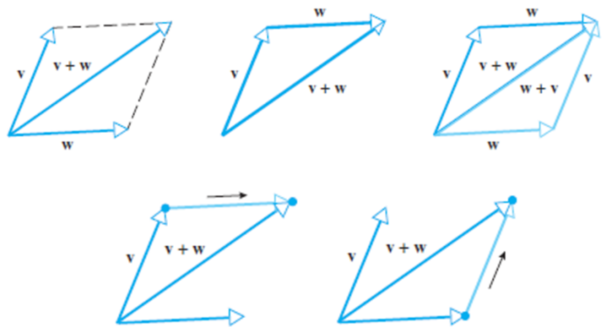
mid 2: Ch2 + Ch3

Let A be a 5x5 matrix and $Ax = 0$ has infinitely many solutions.

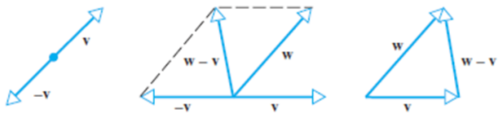
- A is invertible F
- RREF of A is I_5 F
- $\det \neq 0$ F
- At least one eigenvalue is 0 True
- $Ax = b$ has a unique solution False
- trace = 5 can't be determined
- $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \cdot \lambda_5 = 7$ False

Engineers and physicists distinguish between two types of physical quantities—***scalars***, which are quantities that can be described by a numerical value alone, and ***vectors***, which are quantities that require both a number and a direction for their complete physical description. For example, temperature, length, and speed are scalars because they can be fully described by a number that tells "how much"—a temperature of 20°C, a length of 5 cm, or a speed of 75 km/h. In contrast, velocity and force are vectors because they require a number that tells "how much" and a direction that tells "which way"—say, a boat moving at 10 knots in a direction 45° northeast, or a force of 100 lb acting vertically. Although the notions of vectors and scalars that we will study in this text have their origins in physics and engineering, we will be more concerned with using them to build mathematical structures and then applying those structures to such diverse fields as genetics, computer science, economics, telecommunications, and environmental science.

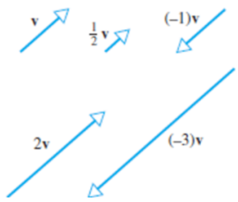
Vectors: Addition of vectors by the parallelogram or triangle rules



Subtraction



Multiplication by a scalar



Vectors Whose Initial Point Is Not at the Origin

It is sometimes necessary to consider vectors whose initial points are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1) \tag{4}$$

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point. For example, in Figure 3.1.12 the vector $\overrightarrow{P_1P_2}$ is the difference of vectors $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$, so

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$$

As you might expect, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \tag{5}$$

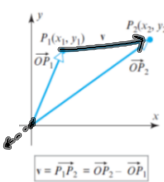


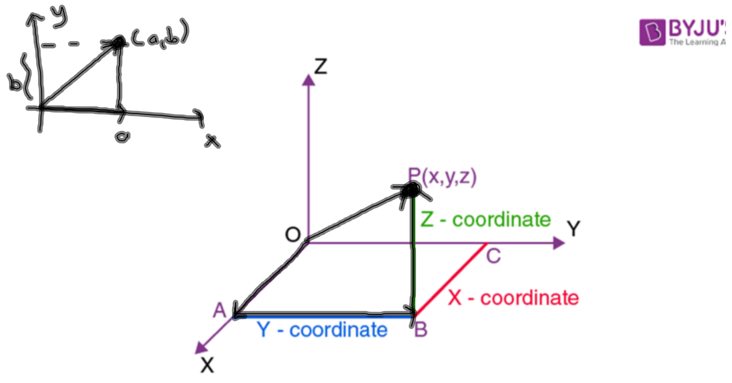
Figure 3.1.12

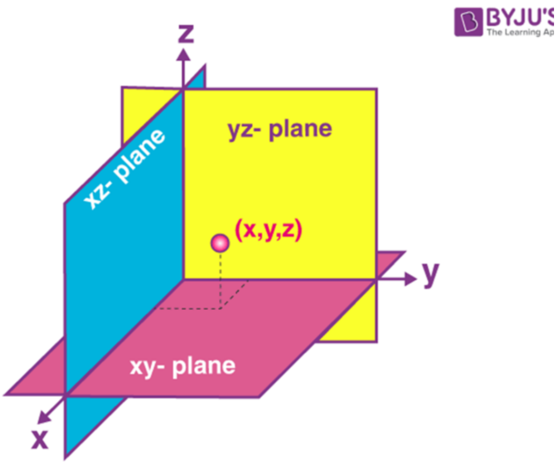
$\overrightarrow{AB} = (x_1 - 0, y_1 - 0) = (x_1, y_1)$

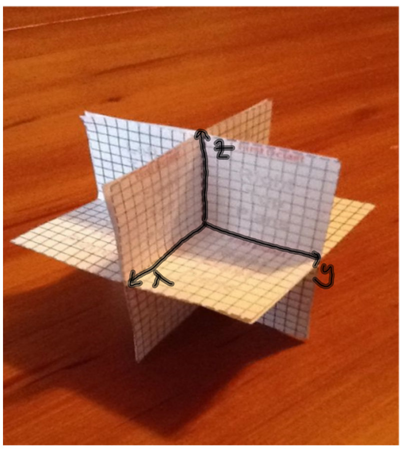
▶ **EXAMPLE 1 Finding the Components of a Vector**

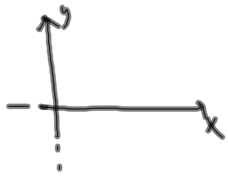
The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

$$\mathbf{v} = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12)$$ ✓

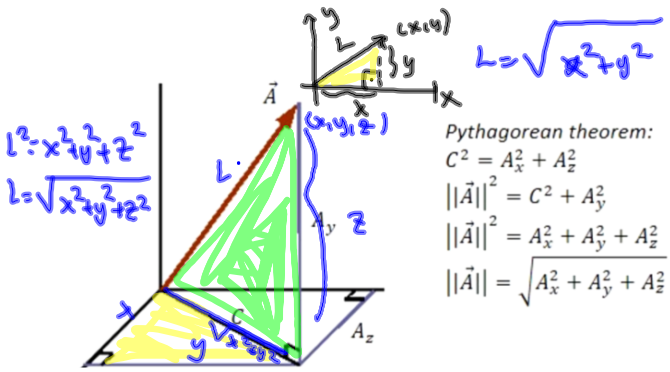








Similar to the two-dimensional coordinat system, here also the point of intersectio of these three axes is the origin O, and these axes divide the space into **eight octants.**



Pythagorean theorem:

$C^2 = A_x^2 + A_z^2$

$||\vec{A}||^2 = C^2 + A_y^2$

$||\vec{A}||^2 = A_x^2 + A_y^2 + A_z^2$

$||\vec{A}|| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Norm

- If $v = (x_1, x_2, \dots, x_n)$ is a vector of R^n then then the norm of v (also called the length of v or the magnitude of v) is denoted by $\|v\|$, is defined by the formula

$$\|v\| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$$

- Example: $v = (1, -0.5, 3)$, calculate the norm of v

  Answer:

$$\|v\| = \sqrt{(1)^2 + (-0.5)^2 + (3)^2} = 3.2016$$

THEOREM 3.2.1 *If* **v** *is a vector in* R^n, *and if k is any scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$ ✓

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ ✓

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

Unite vector

- A unit vector u is a vector with norm equal to one

$$\|u\| = 1$$

- Normalized vector v is a vector u divided by its norm to get a unit vector

$$\|v\| = \frac{1}{\|u\|} u$$

$\left\| \frac{1}{\|u\|} u \right\| = \frac{1}{\|u\|} \|u\| = 1$

---

example: Let $u = (1, 2, 0)$, $v = (-1, 0, 0)$.
which one is False.

✓ a) v is a unit vector

✓ b) $\|u\| = \sqrt{5}$

✓ c) $u + v = (0, 2, 0)$

✓ d) $\|-2u\| = 2\|u\|$

F (e) u is a unit vector

✓ f) $\frac{1}{\|u\|} u$ is unit vector

$\|v\| = \sqrt{(-1)^2 + 0^2 + 0^2} = 1$

$\|u\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

$\left\| \left(\frac{1}{\|u\|}\right) u \right\| = \frac{1}{\|u\|} \cdot \|u\| = 1$

Distance

If $p_1 = (x_1, x_2, \dots, x_n)$ and $p_2 = (y_1, y_2, \dots, y_n)$ are two points of R^n then to calculate the distance between p_1 and p_2 we form the vector $w = \overrightarrow{p_1 p_2}$

Then

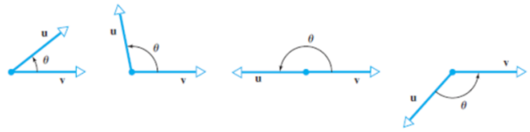
$$dis(p_1, p_2) = \|w\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

mcq: Let $A = (1, -1, 0, -2)$, $B = (-2, -1, 0, -1)$.
distance(A, B) = ?

a) $\sqrt{8}$ (b) $\sqrt{10}$ c) $\sqrt{12}$ d) $\sqrt{6}$ e) $\sqrt{2}$

distance $= \sqrt{3^2 + 0^2 + 0^2 + (1)^2} = \sqrt{10}$

Dot Product



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \le \theta \le \pi$.

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

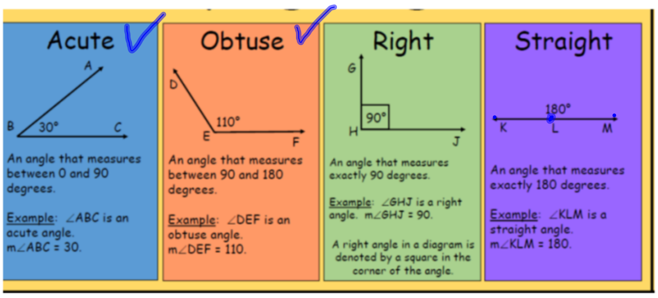
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

Since $0 \le \theta \le \pi$, it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.



DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{17}$$

$$\begin{aligned} 360^\circ &= 2\pi \text{ radians} \\ 1^\circ &= \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians} \\ 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ \end{aligned}$$

Example

a) Convert 65° to radians. b) Convert 1.75 radians to degrees.

Solution

a)

$$\begin{aligned} 1^\circ &= \frac{\pi}{180} \text{ radians} \\ 65^\circ &= 65 \times \frac{\pi}{180} \\ &= 1.134 \text{ radians} \end{aligned}$$

b)

$$\begin{aligned} 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \\ 1.75 \text{ radians} &= 1.75 \times \frac{180}{\pi} \\ &= 100.268^\circ \end{aligned}$$

For $\mathbf{u} = \langle 3, -1, 2\rangle$, $\mathbf{v} = \langle -4, 0, 2\rangle$, $\mathbf{w} = \langle 1, -1, -2\rangle$, and $\mathbf{z} = \langle 2, 0, -1\rangle$, find the angle between each pair of vectors.

a. **u** and **v** **b.** **u** and **w** **c.** **v** and **z** (exercise)

a) $\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{-8}{\sqrt{14} \cdot \sqrt{20}} = -0.47$

$u \cdot v = -12 + 0 + 4 = -8$

θ is obtuse.

$\|u\| = \sqrt{9+1+4} = \sqrt{14}$

$\|v\| = \sqrt{16+0+4} = \sqrt{20}$

$\cos\theta = -0.47$

$\theta = \arccos(-0.47)$

$\theta \cong 2$ radians

$\cong 118$ degrees

b) $u \cdot w = 3 + 1 + -4 = 0$

$\theta = \frac{\pi}{2} = 90^\circ$

$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{0}{?} = 0$

- Two nonzero vectors u and v in R^n are said to be orthogonal (or perpendicular) if their dot product is zero

$$u.v = 0$$

- ~~A nonempty set of vectors in R^n is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.~~

- ~~An orthogonal set of unit vectors is called an orthonormal set.~~

Lines and Planes Determined by Points and Normals

One learns in analytic geometry that a line in R^2 is determined uniquely by its slope and one of its points, and that a plane in R^3 is determined uniquely by its "inclination" and one of its points. One way of specifying slope and inclination is to use a *nonzero* vector $\mathbf{n}$, called a ***normal***, that is orthogonal to the line or plane in question. For example, Figure 3.3.1 shows the line through the point $P_0(x_0, y_0)$ that has normal $\mathbf{n} = (a, b)$ and the plane through the point $P_0(x_0, y_0, z_0)$ that has normal $\mathbf{n} = (a, b, c)$. Both the line and the plane are represented by the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0 \tag{1}$$

where P is either an arbitrary point (x, y) on the line or an arbitrary point (x, y, z) in the plane. The vector $\overrightarrow{P_0P}$ can be expressed in terms of components as

$$\overrightarrow{P_0P} = (x - x_0, y - y_0) \quad \textbf{[line]}$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0) \quad \textbf{[plane]}$$

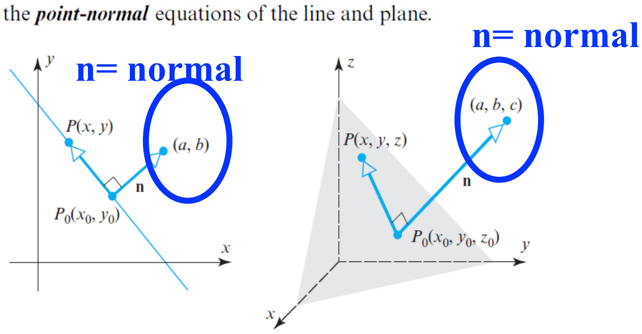
Formula (1) is called the ***point-normal*** form of a line or plane and Formulas (2) and (3) the ***component*** forms.

Thus, Equation (1) can be written as

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]} \tag{2}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]} \tag{3}$$

These are called the ***point-normal*** equations of the line and plane.



▶ Figure 3.3.1

==

▶ EXAMPLE 2 Point-Normal Equations

It follows from (2) that in R^2 the equation

$$6(x - 3) + (y + 7) = 0$$

represents the line through the point $(3, -7)$ with normal $\mathbf{n} = (6, 1)$; and it follows from (3) that in R^3 the equation

$$4(x - 3) + 2y - 5(z - 7) = 0$$

represents the plane through the point $(3, 0, 7)$ with normal $\mathbf{n} = (4, 2, -5)$. ◀

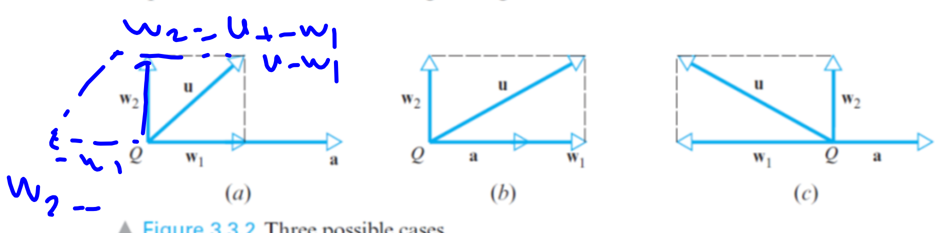
==

- Drop a perpendicular from the tip of **u** to the line through **a**.
- Construct the vector $\mathbf{w}_1$ from Q to the foot of the perpendicular.
- Construct the vector $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

Since

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$$

we have decomposed **u** into a sum of two orthogonal vectors, the first term being a scalar multiple of **a** and the second being orthogonal to **a**.



▲ Figure 3.3.2 Three possible cases.

The following theorem shows that the foregoing results, which we illustrated using vectors in R^2, apply as well in R^n.

THEOREM 3.3.2 Projection Theorem

If **u** *and* **a** *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* **u** *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* **a** *and* $\mathbf{w}_2$ *is orthogonal to* **a**.

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the Projection Theorem have associated names—the vector $\mathbf{w}_1$ is called the ***orthogonal projection of u on a*** or sometimes ***the vector component of* u *along* a**, and the vector $\mathbf{w}_2$ is called the ***vector component of u orthogonal to a***. The vector $\mathbf{w}_1$ is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\mathbf{u}$, in which case it follows from (8) that $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$. In summary,

$$w_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a}) \tag{10}$$

$$w_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a}) \tag{11}$$

$\|u\|\cdot\|a\|\cdot\cos\theta$

$\|w_1\| = \text{Proj}_a u = \left\| \frac{u\cdot a}{\|a\|^2} a \right\|$

$= \frac{\|u\|\cdot\|a\|\cdot\cos\theta \, \|a\|}{\|a\|^2}$

$= \|u\|\cdot\cos\theta$

$\|w_1\| = \|u\|\cdot\cos\theta$

$\|w_2\| = \|u\|\cdot\sin\theta$

$\|w_1\|^2 + \|w_2\|^2 = \|u\|^2$

$\|u\|^2\cos^2\theta + \|w_2\|^2 = \|u\|^2$

$\|w_2\|^2 = \|u\|^2 - \|u\|^2\cos^2\theta$

$\|u\|^2(1-\cos^2\theta)$ $\sin^2\theta$

$\|w_2\| = \|u\|\cdot\sin\theta$

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a**.

$u\cdot a = 8+1+6 = 15$

$\|a\| = \sqrt{16+1+4} = \sqrt{21}$

$w_1 = \text{Proj}_a u = \frac{u\cdot a}{\|a\|^2} a$

$= \frac{15}{21}(4,-1,2) = \left(\frac{60}{21}, \frac{-15}{21}, \frac{30}{21}\right)$

$w_1 = \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)$

$w_2 = u - w_1 = (2,-1,3) - \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)$ $\quad 2-\frac{20}{7} = -\frac{6}{7}$

$= \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$ $\quad -1+\frac{5}{7} \quad 3-\frac{10}{7} =$

Check: w_1 and w_2 are orthogonal

$w_1\cdot w_2 = \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)\cdot\left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$ ✓

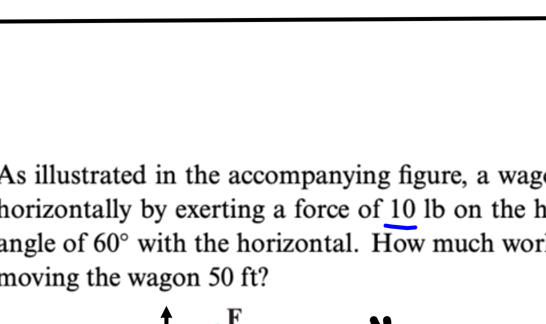
▶ **Exercises 35–37** In physics and engineering the ***work*** W performed by a *constant force* **F** applied in the *direction of motion* to an object moving a distance d on a straight line is defined to be

$$W = \|\mathbf{F}\|d \quad \text{(force magnitude times distance)}$$

In the case where the applied force is constant but makes an angle θ with the direction of motion, and where the object moves along a line from a point P to a point Q, we call $\overrightarrow{PQ}$ the ***displacement*** and define the work performed by the force to be

$$W = \mathbf{F}\cdot\overrightarrow{PQ} = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos\theta$$

(see accompanying figure). Common units of work are ft-lb (foot pounds) or Nm (Newton meters). ◀



$$\text{Work} = (\|\mathbf{F}\|\cos\theta)\|\overrightarrow{PQ}\|$$

lel

36. As illustrated in the accompanying figure, a wagon is pulled horizontally by exerting a force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?

nd

$W = \|F\|\cdot\cos\theta\cdot 50$

$= 10\cdot\cos 60\cdot 50$ $\quad 1/2$

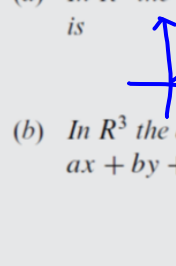
$= 250$

Step 1 1 of 2

We have $\|\mathbf{F}\| = 10$ lb, $\|\vec{d}\| = 50$ ft and $\theta = 60°$, where θ is the angle between **F** and $\vec{d}$. Therefore,

$$W = \|\mathbf{F}\|\|\vec{d}\|\cos\theta = (10)(50)\cos(60°) = (500)\frac{1}{2} = 250$$

The Theorem of Pythagoras



In Section 3.2 we found that many theorems about vectors in R^2 and R^3 also hold in R^n. Another example of this is the following generalization of the Theorem of Pythagoras (Figure 3.3.5).

THEOREM 3.3.3 Theorem of Pythagoras in R^n

If **u** *and* **v** *are orthogonal vectors in* R^n *with the Euclidean inner product, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{14}$$ ✓

(a) *In* R^2 *the distance D between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(b) *In* R^3 *the distance D between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

ex. Distance between $x+y=-2$, $(2,7)$

$1\cdot x + 1\cdot y + 2 = 0$ $\quad (2,7)$
a b c $\quad x_0\ y_0$

$D = \frac{|1\cdot 2 + 1\cdot 7 + 2|}{\sqrt{1^2+1^2}} = \frac{11}{\sqrt{2}}$.

ex. Distance $2x + y = -3$, $(-1,-1)$

$2x + 1y + 3 = 0$ $\quad (-1,-1)$
a b c $\quad x_0\ y_0$

$D = \frac{|2\cdot -1 + 1\cdot -1 + 3|}{\sqrt{2^2+1^2}} = \frac{0}{\sqrt{5}} = 0$.

Because $(-1,-1)$ is on the line $2x+y=-3$

Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

$x_0\ y_0\ z_0$: $1, -4, -3$

$2x - 3y + 6z + 1 = 0$
a b c d

$D = \frac{|2 + 12 + 18 + 1|}{\sqrt{4+9+36}} = \frac{3}{7}$.

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \checkmark$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \checkmark \quad (1)$$

Dot product is a number, but cross product is a vector.

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$.

$U \times V = (2, -7, -6)$ ✓ $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ✓ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$] ✓

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ✓ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$] ✓

(c) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ [*Lagrange's identity*] ✓

(d) ~~$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ [*vector triple product*]~~

(e) ~~$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ [*vector triple product*]~~

mcq: Let U and V are two vectors in $\mathbb{R}^3$. Which one is False

a) $U \cdot (U \times V) = 0$ ✓

b) if $U \cdot V = 0$, U and V are orthogonal ✓

c) V is orthogonal to $U \times V$ ✓

d) if U and V are orthogonal, $\|U + V\|^2 = \|U\|^2 + \|V\|^2$ ✓

e) $U \times V$ is a number ✗

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ ✓

(b) ~~$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$~~

(c) ~~$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$~~

(d) ~~$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$~~

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$ ✓

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ ✓

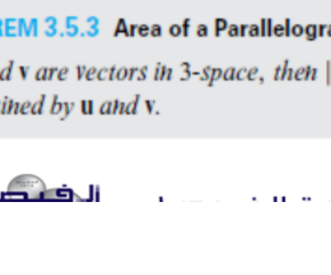
$U \cdot V = \|U\| \cdot \|V\| \cdot \cos\theta$

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

$\|U \times V\|^2 = \|U\|^2\|V\|^2 - (\|U\|^2\|V\|^2\cos^2\theta)$

$= \|U\|^2\|V\|^2(1 - \cos^2\theta)$, where $1 - \cos^2\theta = \sin^2\theta$

$\|U \times V\| = \|U\| \cdot \|V\| \cdot \sin\theta$



$A = \text{(base)(altitude)} = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then* $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

▶ EXAMPLE 4 Area of a Triangle

Find the area of the triangle determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

$\overrightarrow{P_1P_2} = (-3, -2, 2)$

$\overrightarrow{P_1P_3} = (-2, 2, 3)$

$\begin{vmatrix} -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$

▲ Figure 3.5.5

$P_1P_2 \times P_1P_3 = (-10, 5, -10)$

$\|P_1P_2 \times P_1P_3\| = \sqrt{100 + 25 + 100} = \sqrt{225} = 15$

Triangle: $\frac{15}{2} = 7.5$

mid2: Ch2 + Ch3 SLIDES(1–46) not included

$17 \times 1.5 = 25.5$

Collect max: 22

Definition (linear combination)

If w is a vector in a vector space V, then w is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_r$ in V if w can be expressed in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the *coefficients* of the linear combination

---

EXAMPLE 14 Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$(9,2,7) = a\cdot(1,2,-1) + b(6,4,2)$

$(9,2,7) = (a, 2a, -a) + (6b, 4b, 2b)$

$(9,2,7) = (a+6b, 2a+4b, -a+2b)$

$\begin{cases} a+6b = 9 \\ 2a+4b = 2 \\ -a+2b = 7 \end{cases}$ $\quad \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix}$

$-8b = -16$

$b = 2$

$a + 6b = 9$, $6b = 12$

$a = -3$

$(9,2,7) = -3(1,2,-1) + 2\cdot(6,4,2)$

$(4,-1,8) = a(1,2,-1) + b(6,4,2)$

$\begin{cases} a+6b = 4 \\ 2a+4b = -1 \\ -a+2b = 8 \end{cases}$ $\quad \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix}$

no solution $\leftarrow$ X $\begin{cases} -8b = -9 \\ 8b = 12 \end{cases}$

- **Definition** Let $S = \{v_1, v_2, \ldots, v_r\}$, be a nonempty set of vectors of a vector space V. And consider the equation

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$$

a trivial solution is $k_1 = k_2 = \cdots = k_r = 0$

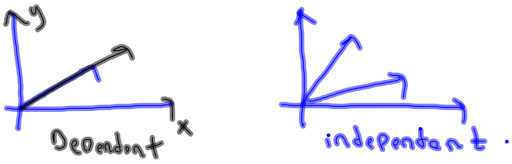
- If the trivial solution is the only solution, then S is said to be a linearly independent set
- If there are solutions in addition to the trivial solution, then S is said to be a linearly dependent set.

Geometrically:

Let u and v be 2 vectors and dependent.

$c_1 u + c_2 v = 0$, $c_1 \neq 0$

$c_1 u = -c_2 v \Rightarrow u = \left(-\frac{c_2}{c_1}\right) v = c v$



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mcq: which ones are dependents?

- ~~(a)~~ $u=(1,2),\ v=(2,3)$
- ~~(b)~~ $u=(1,1),\ v=(2,3)$
- ~~(b)~~ $u=(0,1),\ v=(4,3)$
- ~~(d)~~ $u=(5,1),\ v=(1,5)$
- **(c)** $u=(-1,3),\ v=(2,-6)$ — $v=-2\cdot u$

Let's say u, v, w are dependent.

$$c_1u+c_2v+c_3w=0,\quad c_1\neq 0$$

$$c_1u=-c_2v-c_3w$$

$$u=\underbrace{\left(-\frac{c_2}{c_1}\right)}_{A}v\underbrace{\left(-\frac{c_3}{c_1}\right)}_{B}w \Rightarrow u=Av+Bw$$

u is linear combination of v and w.

Determine whether the vectors

$$\mathbf{v}_1=(1,-2,3),\quad \mathbf{v}_2=(5,6,-1),\quad \mathbf{v}_3=(3,2,1)$$

are linearly independent or linearly dependent in R^3.

$$c_1v_1+c_2v_2+c_3v_3=(0,0,0)$$

$$c_1(1,-2,3)+c_2(5,6,-1)+c_3(3,2,1)=(0,0,0)$$

$$\begin{cases} c_1+5c_2+3c_3=0\\ -2c_1+6c_2+2c_3=0\\ 3c_1-c_2+c_3=0\end{cases}$$

→ det
→ elimination

elimination:

$$\begin{bmatrix}1&5&3&0\\-2&6&2&0\\3&-1&1&0\end{bmatrix}\rightarrow\begin{bmatrix}1&5&3&0\\0&16&8&0\\0&-16&-8&0\end{bmatrix}$$

$$\begin{bmatrix}1&5&3&0\\0&1&1/2&0\\0&-16&-8&0\end{bmatrix}\rightarrow\begin{bmatrix}1&5&3&0\\0&1&1/2&0\\0&0&0&0\end{bmatrix}$$

$$\begin{cases} c_1+5c_2+3c_3=0\\ c_2+\frac{c_3}{2}=0\end{cases}$$

Dependent vectors. Because, inf. many solutions

$$c_2=-\frac{1}{2}c_3 \checkmark$$

$$c_1=-5c_2-3c_3=\frac{5}{2}c_3-3c_3=-\frac{1}{2}c_3$$

$$\begin{cases} c_1=-\frac{1}{2}c_3\\ c_2=-\frac{1}{2}c_3\\ c_3 \text{ free}\end{cases}\rightarrow\begin{cases} c_1=1\\ c_2=1\\ c_3=-2\end{cases}$$

$$c_1v_1+c_2v_2+c_3v_3=0$$

$$1v_1+1v_2-2v_3=0$$

$$v_1=-v_2+2v_3$$

determinant: $\begin{cases} Ax=0 \text{ has only zero solution} \\ \det A \neq 0 \end{cases}$

$$\det = \begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} \overset{\text{row } 1}{=} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 1\cdot(8) + 5(8) + 3(-16) = 0$$

$\det = 0 \Rightarrow$ infinitely many solutions $\Rightarrow$ dependent.

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

$c_1(1,2,2,-1) + c_2(4,9,9,-4) + c_3(5,8,9,-5) = 0$

$$\begin{cases} c_1 + 4c_2 + 5c_3 = 0 \\ 2c_1 + 9c_2 + 8c_3 = 0 \\ 2c_1 + 9c_2 + 9c_3 = 0 \\ -c_1 - 4c_2 - 5c_3 = 0 \end{cases} \Rightarrow \det \text{ Fails}$$

Coef. mat $= \begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & \textcircled{1} & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to c_1 + 4c_2 + 5c_3 = 0 \Rightarrow c_1 = 0 \\ \to c_2 = 2c_3 = 0 \\ \to c_3 = 0 \end{array}$$

We have only zero solution

v_1, v_2, v_3 are independent.

THEOREM 4.3.2

(a) ✓ *A finite set that contains* **0** *is linearly dependent.*

(b) ✓ *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

(c) ✓ *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

(b) $S = \{u\}$ $\quad c \cdot u = 0 \;\exists\, c \in \mathbb{R}$ if $u = 0$

(a) $S = \{u, v, w, 0\} \to$ dependent.

$c_1 u + c_2 v + c_3 w + c_4 0 = 0$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 0, \quad c_4 = 1, \; c_4 = 2, \; c_4 = 3, \ldots$

---

2 Rank of a Matrix

Let A be an $m \times n$ matrix.

Definition 3. *The* **rank** *of a matrix* **A***—denoted as rank* **A***—is the maximum number of linearly independent row vectors of* **A**.

nullity of a matrix.

nullity $= n - \text{Rank}(A)$

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n \tag{4}$$

n → number of columns

- *How to calculate the Rank and the Nullity*
- *Answer:*
- Perform a Gauss-Elimination to *Get the row-echelon form of A* and *count the number of leader 1.*

Rank = number of leader 1.

Nullity = number of free variables

Important Remark:

- *Rank(A) = number of leader 1 in the row echelon form of A = number of ~~dependant~~ basic variables*
- *nullity(A) = number of free variables = number of parameter in the general solution of the homogenous system*

THEOREM 4.8.8 Equivalent Statements

If A is an n × n matrix, then the following statements are equivalent.

(a) *A is invertible.* Ch 1

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.* Ch 1

(c) *The reduced row echelon form of A is* I_n. Ch 1

(d) ~~*A is expressible as a product of elementary matrices.*~~

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every n × 1 matrix* **b**. Ch 1

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every n × 1 matrix* **b**. Ch 1

(g) $\det(A) \neq 0$. Ch 2

(h) *The column vectors of A are linearly independent.* Ch 3

(i) *The row vectors of A are linearly independent.* Ch 3

(j) ~~*The column vectors of A span* R^n.~~

(k) ~~*The row vectors of A span* R^n.~~

(l) ~~*The column vectors of A form a basis for* R^n.~~

(m) ~~*The row vectors of A form a basis for* R^n.~~

(n) *A has rank n.* Ch 3

(o) *A has nullity 0.* Ch 3

(s) $\lambda = 0$ is ~~not~~ an eigenvalue Ch 5

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}_{3\times 3}$$

Rank + nullity = 3.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Nullity = 1

What you can say for A:

$\lambda = 0$ is an eigenvalue

Row vectors are dependent

column vectors are dependent

A is not invertible

$\det A = 0$

$Ax = b$ can't have a unique solution

$Ax = 0$ must have inf. many.

RREF can't be $I_{3\times 3}$.

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}_{4\times 5}$$

Rank + nullity = 5

$$\to \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

nullity = 3

Which one is True/False

- Row vectors are independent — False

- $Ax = 0$ must have only zero solution — False

Let A be a 5×5 matrix and nullity = 1

Which one is True.

~~a) Rank = 1~~ Rank = 4

b) Row vectors are dependent — True

c) $\det A \neq 0$ — False

d) A is invertible — False

e) $\lambda = 0$ must be an eigenvalue. — True

Let A be a 3×3 matrix and its Characteristic polynomial is $P(\lambda) = (\lambda - 2)^2 \cdot (\lambda + 1)$. Then which one is True/False

$(\lambda-2)^2(\lambda+1) = 0$

$(\lambda-2)(\lambda-2)(\lambda+1) = 0$

$\lambda = 2 \quad \lambda = 2 \quad \lambda = -1$

F – $\lambda = 0$ is an eigenvalue

T – Rank = 3

T – $Ax = 0$ must have only zero solution

T – nullity = 0

F – $\det = 0$

T – in $Ax = 0$, we have no free variables

T – In RREF, there are 3 leader 1

Next week: Ch 4

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considering such inequalities in general.

Definition

A *linear inequality* in the variables x and y is an inequality that can be written in one of the forms

$$ax + by + c < 0 \quad ax + by + c \leq 0 \quad ax + by + c > 0 \quad ax + by + c \geq 0$$

where a, b, and c are constants and not both a and b are zero.

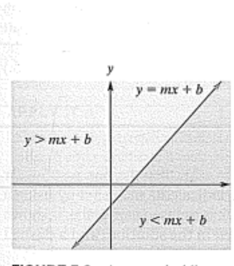

FIGURE 7.2 A nonvertical line determines two half-planes.

Solve the inequality $2(2x - y) < 2(x + y) - 4$.

$4x - 2y < 2x + 2y - 4$

$4x - 2x + 4 < 2y + 2y$

$2x + 4 < 4y \Rightarrow \frac{x}{2} + 1 < y$

$\Rightarrow y > \frac{x}{2} + 1$

$y = \frac{x}{2} + 1$

Now Work Problem 1 ◁

Systems of Inequalities

The solution of a *system* of inequalities consists of all points whose coordinates simultaneously satisfy all of the given inequalities. Geometrically, it is the region that is common to all the regions determined by the given inequalities. For example, let us solve the system

$$\begin{cases} 2x + y > 3 \\ x \geq y \\ 2y - 1 > 0 \end{cases}$$

We first rewrite each inequality so that y is isolated. This gives the equivalent system

$$\begin{cases} y > -2x + 3 \\ y \leq x \\ y > \frac{1}{2} \end{cases}$$

Next, we sketch the corresponding lines $y = -2x + 3$, $y = x$, and $y = \frac{1}{2}$, using dashed lines for the first and third and a solid line for the second. We then shade the region that is below the first line, the region that is above the second line, and the region that is below the third line. The region that is unshaded (Figure 7.9) together with any solid line boundaries are the points in the solution of the system of inequalities.

EXAMPLE 3 **Solving a System of Linear Inequalities**

Solve the system

$$\begin{cases} y \geq -2x + 10 \\ y \geq x - 2 \end{cases}$$

Solution: The solution consists of all points that are simultaneously on or above the line $y = -2x + 10$ and on or above the line $y = x - 2$. It is the unshaded region in Figure 7.10.

Now Work Problem 9 ◁

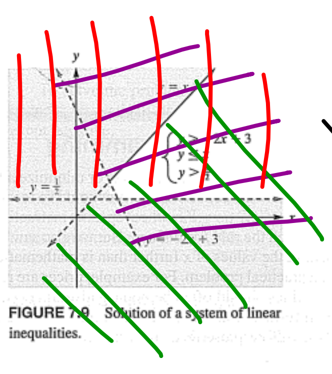

FIGURE 7.9 Solution of a system of linear inequalities.

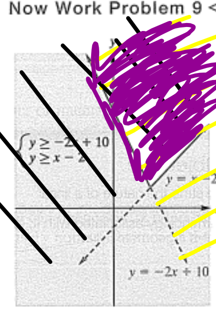

FIGURE 7.10 Solution of a system of linear inequalities.

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variables and, hence, a wider range of problems.

We consider the following problem. A company produces two types of can openers: manual and electric. Each requires in its manufacture the use of three machines: A, B, and C. Table 7.1 gives data relating to the manufacture of these can openers. Each manual can opener requires the use of machine A for 2 hours, machine B for 1 hour, and machine C for 1 hour. An electric can opener requires 1 hour on A, 2 hours on B, and 1 hour on C. Furthermore, suppose the maximum numbers of hours available per month for the use of machines A, B, and C are 180, 160, and 100, respectively. The profit on a manual can opener is \$4, and on an electric can opener it is \$6. If the company can sell all the can openers it can produce, how many of each type should it make in order to maximize the monthly profit?

Table 7.1

	Manual (x)	Electric (y)	Hours Available
A	2 hr	1 hr	180
B	1 hr	2 hr	160
C	1 hr	1 hr	100
Profit/Unit	\$4	\$6	

$$\begin{cases} 4x + 6y = P(max) \\ 2x + y \leq 180 \\ x + 2y \leq 160 \\ x + y \leq 100 \\ x \geq 0, y \geq 0 \end{cases}$$

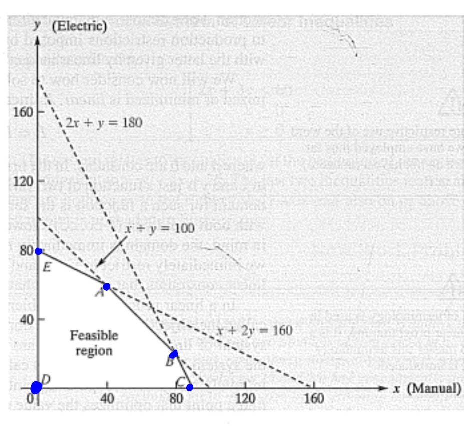


FIGURE 7.12 Feasible region.

a maximum value of P) and that has at least one point in common with the region. It is not difficult to observe that such a line will contain the corner point isoprofit line with a greater profit will contain no points of the feasible region.

From Figure 7.12, we see that A lies on both the line $x + y = 100$ and $x + 2y = 160$. Thus, its coordinates may be found by solving the system

$$\begin{cases} x + y = 100 \\ x + 2y = 160 \end{cases}$$

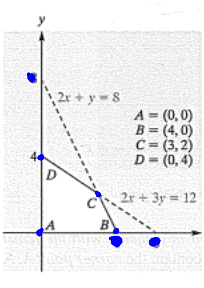
This gives $x = 40$ and $y = 60$. Substituting these values into the equation $P = \ldots$ we find that the maximum profit subject to the constraints is \$520, which is by producing 40 manual can openers and 60 can openers per month.

If a feasible region can be contained within a circle, such as the region in Fig... it is called a **bounded feasible region.** Otherwise, it is **unbounded.** When a region contains at least one point, it is said to be **nonempty.** Otherwise, it i... The region in Figure 7.13 is a nonempty, bounded feasible region.

It can be shown that

It can be shown that

A linear function defined on a nonempty, bounded feasible region has a maximum (minimum) value, and this value can be found at a corner point.



EXAMPLE 1 **Solving a Linear Programming Problem**

Maximize the objective function $P = 3x + y$ subject to the constraints

$$2x + y \leq 8$$
$$2x + 3y \leq 12$$
$$x \geq 0$$
$$y \geq 0$$

$P(x,y) = 3x + y$

Solution: In Figure 7.14, the feasible region is nonempty and bounded. Thus, P is maximum at one of the four corner points. The coordinates of A, B, and D are obvious on inspection. To find the coordinates of C, we solve the equations $2x + y = 8$ and $2x + 3y = 12$ simultaneously, which gives $x = 3$, $y = 2$. Thus,

A) $(0,0)$: $P(A) = 3 \cdot 0 + 0 = 0$ ✓ (min)

B) $y = 0 \Rightarrow 2x = 8 \Rightarrow x = 4$ $B = (4,0)$:

$P(B) = 3 \cdot 4 + 0 = 12$ ✓ (max)

C) $-\ 2x + y = 8$
$2x + 3y = 12$
$2y = 4$
$y = 2$

$\rightarrow 2x + 6 = 12$
$2x = 6$
$x = 3$

$C = (3,2)$

$P(C) = P(3,2) = 3 \cdot 3 + 2 = 11$ ✓

D) $2 \cdot 0 + 3y = 12 \Rightarrow y = 4$

$D = (0,4)$

$P(x,y) = 3x + y$

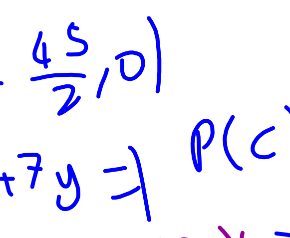
$P(D) = 3 \cdot 0 + 4 = 4$ ✓

---

1. Maximize $P = 5x + 7y$

subject to

$2x + 3y \leq 45$
$x - 3y \geq 2$
$x, y \geq 0$



$x - 2 \geq 3y$ $\frac{x-2}{3} \geq y$

$y \leq \frac{x-2}{3}$

$3y \leq 45 - 2x$

A: $x - 3y = 2$: set $y = 0 \Rightarrow x = 2$

A: $(x,y) = (2,0)$ $P(A) = 10$ ✓ (min)

C: $2x + 3y = 45$: set $y = 0$; $2x = 45$, $x = 45/2$

$C = (\frac{45}{2}, 0)$

$P = 5x + 7y \Rightarrow P(C) = 5 \cdot \frac{45}{2} + 0$

max. $= \frac{225}{2} = 112.5$

B: $2x + 3y = 45$
$+ \ x - 3y = 2$
$3x = 47 \Rightarrow x = \frac{47}{3}$

$\rightarrow \frac{47}{3} - 3y = 2$

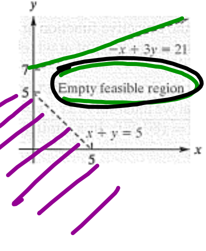
$\frac{47}{3} - 2 = 3y$

$\frac{41}{3} = 3y$

$y = \frac{41}{9}$

$P(B) = 5 \cdot \frac{47}{3} + 7 \cdot \frac{41}{9}$

$= 110$.



EXAMPLE 2 **Empty Feasible Region**

Minimize the objective function $Z = 8x - 3y$, subject to the constraints

$$-x + 3y = 21$$
$$x + y \leq 5$$
$$x \geq 0$$
$$y \geq 0$$

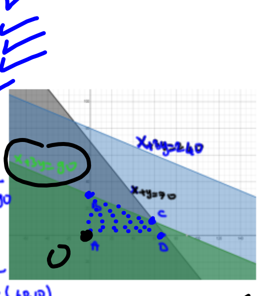
Whenever the feasible region of a linear programming problem is empty, no optimum solution exists.

2. Maximize

$$P = 3x + 2y$$

subject to

$$x + y \le 70$$
$$x + 3y \le 240$$
$$x + 3y \le 90$$
$$x, y \ge 0$$



min.

$P = 3x + 2y$

A(0,0) minimum

$P(A) = 3\cdot0 + 2\cdot0 = 0$

B y-intercept of $x+3y=90$

$x=0$: $y=30$

B(0,30)

$P(B) = 3\cdot0 + 2\cdot30 = 60$

C: $x+y=70$
$+\ x+3y=90$
$2y=20$
$y=10$
$x=60$

C:(60,10)

$P(C) = 3\cdot60 + 2\cdot10 = 200$

Max

D: x-intercept of $x+y=70$ D(70,0)

$y=0$: $x=70$

$P(D) = 3\cdot70 + 2\cdot0 = 210$ Max

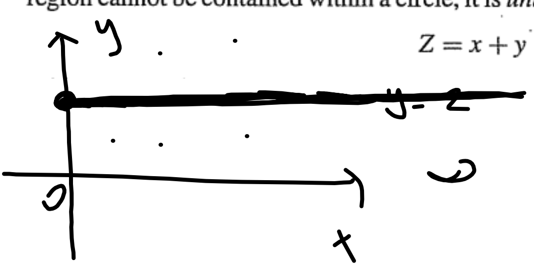
Unbounded Feasible Region

Suppose a feasible region is defined by

Suppose a feasible region is defined by

$$y = 2$$
$$x \geq 0$$
$$y \geq 0$$

This region is the portion of the horizontal line $y = 2$ indicated in Figure 7.16. Since the region cannot be contained within a circle, it is *unbounded*. Let us consider maximizing



$$Z = x + y$$

$Z = x + y$

$Z = x + 2$

maximum: doesn't exist

minimum: $x = 0$, $y = 2$ $\Rightarrow Z = 2$

In general, it can be shown that

If a feasible region is unbounded, and *if* the objective function has a maximum (or minimum) value, then that value occurs at a corner point.

Maximize $P = 2x + 4y$ subject to the constraints

$$x - 4y \leq -8$$
$$x + 2y \leq 16$$
$$x, y \geq 0$$

$P(A) = 8$

$P(B) = 32$

$P(C) = 32$

$A: (0,2)$

$B: (8,4)$

$C: (0,8)$

$x - 4y = -8$

$x + 2y = 16$

$6y = 24$

$y = 4$

$x = 8$

$x = 0$

$-4y = -8$

$y = 2$

$x = 0$

$2y = 16$ $y = 8$

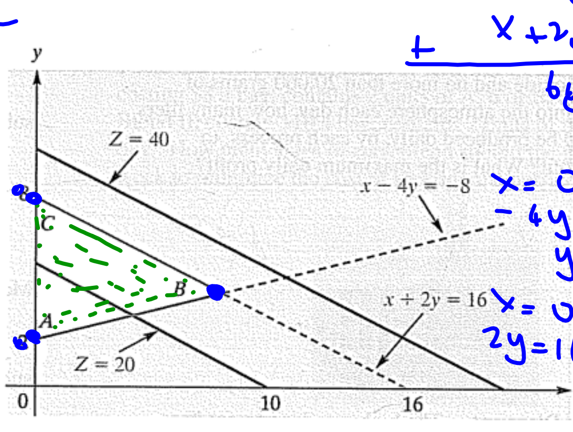


FIGURE 7.18 $Z = 2x + 4y$ is maximized at each point on the line segment $\overline{BC}$

Simplex algorithm

Simplex Method

Problem:

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3$

subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq b_3$$
$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \leq b_4$$

where x_1, x_2, x_3 and b_1, b_2, b_3, b_4 are nonnegative.

Method:

Maximize

$$Z = 2x_1 - x_2 + x_3$$

subject to

$$2x_1 + x_2 - x_3 \leq 4$$
$$x_1 + x_2 + x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

$$\begin{cases} 2x_1 + x_2 - x_3 + S_1 = 4 \\ x_1 + x_2 + x_3 + S_2 = 2 \\ -2x_1 + x_2 - x_3 + Z = 0 \end{cases}$$

entering

departing variable

$$\begin{array}{c} \\ S_1 \\ S_2 \\ Z \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & S_1 & S_2 & Z & R \\ 2 & 1 & -1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ -2 & 1 & -1 & 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{c} \\ 2 \\ 2 \\ \\ \end{array}$$

$$\begin{array}{c} \\ S_1 \\ x_1 \\ Z \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & S_1 & S_2 & Z & R \\ 0 & -1 & -3 & 1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 0 & 2 & 1 & 4 \end{array}$$

$$\begin{cases} Z = 4 \\ x_1 = 2 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Use the simplex method to solve the following problems.

1. Maximize

$$Z = x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 8$$
$$2x_1 + 3x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

$$\begin{cases} 2x_1 + x_2 + S_1 = 8 \\ 2x_1 + 3x_2 + S_2 = 12 \\ -x_1 - 2x_2 + Z = 0 \end{cases}$$

entering

departing

$$\begin{array}{c} \\ S_1 \\ S_2 \\ Z \end{array} \begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & R \\ 2 & 1 & 1 & 0 & 0 & 8 \\ 2 & 3 & 0 & 1 & 0 & 12 \\ -1 & -2 & 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{l} \\ 8:1 = 8 \\ 12:3 = 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ S_1 \\ x_2 \\ Z \end{array} \begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & R \\ 2 & 1 & 1 & 0 & 0 & 8 \\ 2/3 & 1 & 0 & 1/3 & 0 & 4 \\ -1 & -2 & 0 & 0 & 1 & 0 \end{array}$$

$-\frac{2}{3} + 2$

$\frac{4}{3} - 1$

$$\begin{array}{c} \\ S_1 \\ x_2 \\ Z \end{array} \begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & R \\ 4/3 & 0 & 1 & -1/3 & 0 & 4 \\ 2/3 & 1 & 0 & 1/3 & 0 & 4 \\ 1/3 & 0 & 0 & 2/3 & 1 & 8 \end{array}$$

$$\begin{cases} Z = 8 \\ x_2 = 4 \\ x_1 = 0 \end{cases}$$

---

2. Maximize

$$Z = 2x_1 + x_2$$

subject to

$$-x_1 + x_2 \leq 4$$
$$x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

$$\begin{cases} -x_1 + x_2 + S_1 = 4 \\ x_1 + x_2 + S_2 = 6 \\ -2x_1 - x_2 + Z = 0 \end{cases}$$

entering

departing

$$\begin{array}{c} \\ S_1 \\ S_2 \\ Z \end{array} \begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & R \\ -1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 1 & 0 & 6 \\ -2 & -1 & 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{l} \\ \cancel{4:-1=-4} \\ 6 \\ \\ \end{array}$$

$$\begin{array}{c} \\ S_1 \\ x_1 \\ Z \end{array} \begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & R \\ 0 & 2 & 1 & 1 & 0 & 10 \\ 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 2 & 1 & 12 \end{array}$$

$$\begin{cases} Z = 12 \\ x_1 = 6 \\ x_2 = 0 \end{cases}$$

Maximize $Z = 5x_1 + 4x_2$ subject to

$$x_1 + x_2 \le 20$$
$$2x_1 + x_2 \le 35$$
$$-3x_1 + x_2 \le 12$$

$x_1 + x_2 + s_1 = 20$
$2x_1 + x_2 + s_2 = 35$
$-3x_1 + x_2 + s_3 = 12$
$-5x_1 - 4x_2 + Z = 0$

entering: x_1 — departing: s_2

	x_1	x_2	s_1	s_2	s_3	Z	R	
s_1	1	1	1	0	0	0	20	20 : 1 = 20
s_2	2	1	0	1	0	0	35	35 : 2 = 17.5
s_3	−3	1	0	0	1	0	12	~~12 : −3 = −4~~
Z	−5	−4	0	0	0	1	0	

	x_1	x_2	s_1	s_2	s_3	Z	R
s_1	1	1	1	0	0	0	20
x_1	1	1/2	0	1/2	0	0	17.5
s_3	−3	1	0	0	1	0	12
Z	−5	−4	0	0	0	1	0

entering: x_2 — departing: s_1

	x_1	x_2	s_1	s_2	s_3	Z	R	
s_1	0	1/2	1	−1/2	0	0	2.5	5
x_1	1	1/2	0	1/2	0	0	17.5	35
s_3	0	5/2	0	3/2	1	0	129/2	129/5
Z	0	−3/2	0	5/2	0	1	175/2	

x_2 – pivot

Our new table is

B	x_1	x_2	s_1	s_2	s_3	Z	R
x_2	0	1	2	−1	0	0	5
x_1	1	0	−1	1	0	0	15
s_3	0	0	−5	4	1	0	52
Z	0	0	3	1	0	1	95

indicators

$x_2 = 5$ ✓
$x_1 = 15$ ✓
→ $Z = 95$ ✓

9. Maximize

$Z = 2x_1 + x_2 - x_3$

subject to

$x_1 + x_2 \le 1$

$x_1 - 2x_2 - x_3 \ge -2$

$x_1, x_2, x_3 \ge 0$

~~10- Maximize~~

$-x_1 + 2x_2 + x_3 \le 2$

(S1)

$x_1 + x_2 + S_1 = 1$

$-x_1 + 2x_2 + x_3 + S_2 = 2$

$-2x_1 - x_2 + x_3 + z = 0$

(S2)

B	x_1	x_2	x_3	s_1	s_2	z	R
s_1	1	1	0	1	0	0	1
s_2	-1	2	1	0	1	0	2
z	-2	-1	1	0	0	1	0

1

-2X

(S3) x_1 – entering

S_1 – departing

1 – Pivot

(S4)

B	x_1	x_2	x_3	s_1	s_2	z	R
x_1	1	1	0	1	0	0	1
s_2	0	3	1	1	1	0	3
z	0	1	1	2	0	1	2

non-negative

$R_2 \to R_1 + R_2$

$R_3 \to 2R_1 + R_3$

STOP $z = 2$

$x_1 = 1$

$x_2 = x_3 = 0$.

Solve the following LPP using simplex method:

1- Max $Z = 3X_1 + 4X_2$

Subject to

$15X_1 + 10X_2 \le 300$

$2.5X_1 + 5X_2 \le 110$

$X_1 \ge 0, X_2 \ge 0$

entering

	x_1	x_2	S_1	S_2	~~Solution~~ Z	Ratio
S_1	15	10	1	0	300	300/10=30
S_2	2.5	5	0	1	110	110/5=22
Z	-3	-4	0	0	0	

departing

entering

	x_1	x_2	S_1	S_2	~~Solution~~ Z	Ratio
S_1	10	0	1	-2	80	80/10=8
x_2	0.5	1	0	0.2	22	22/0.5=44
Z	-1	0	0	4/5	88	

departing

	x_1	x_2	S_1	S_2	~~Solution~~ Z
x_1	1	0	0.1	-0.2	8
x_2	0	1	-0.05	0.3	18
Z	0	0	0.1	0.6	96

$Z_{max} = 96$ $X_1 = 8$

$X_2 = 18$